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"You're just my type!" Matching and Payoffs when Like Attracts Like

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Abstract: I analyse a marriage market with transferable utility when the output of two matched agents is a decreasing function of the difference in their types i.e. *like attracts like*. The pattern of sorting and payoffs exhibit many features not found in the standard model, where *more is always better*. Full positive sorting occurs not only if the output function is concave (equivalent to supermodularity) but also if the distribution of types is the same on each side of the market, regardless of the technology. Convexity of the output function is not in general equivalent to submodularity and negative sorting occurs only if there is no overlap in the two type distributions; otherwise there is a mix of perfect matching and negative sorting. For both sides of the market, payoffs as a function of type tend to display a wavelike pattern and are only weakly connected to the quality of the match an agent is in. At types where one payoff function is increasing, the other is decreasing. With convexity, we have maximum possible matching of like with exactly like, so for agents on the long side of the market their optimal choice of partner is not unique. Even though like is attracted to like, having a type close to the mean type on the other side does not always imply a high payoff, and when the marriage market is embedded in a wider economy providing outside options such agents may well remain single.

Keywords: Matching; sorting; marriage market; horizontal heterogeneity; homophily; transferable utility.

JEL Classification Number: C7

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1 Introduction

This paper examines a class of matching problems when agents on one side of the market are attracted to agents on the other side with similar characteristics; i.e. *like attracts like.* With such preferences, agents from one side will typically differ in how they rank potential partners from the other side, in contrast to the case where agents may differ in their own characteristics but nevertheless agree on how potential parters should be ranked; i.e. *more is always better.*

There are many situations where it is useful to think of agents as preferring to match with someone who is similar to them, or who fits in with their own objectives or capabilities; such preferences are sometimes called *homophilic.*[1] A person in search of a partner might prefer someone of the same age, or who has similar tastes or political views, or comes from a similar cultural, social, or ethnic background.[2] Potential adopters may prefer a child of their own race, and adoption agencies may prefer to place a child in a family of the same race.[3] Moving beyond the personal, employers may prefer, and be preferred by, workers who share their goals or sense of mission.[4] Doctors may differ on the right way to treat patients, and patients may differ on how they want to be treated.[5] The assignment of interns to hospitals has long been an important application of matching theory; e.g. Roth (1984). A hospital specialising in new cancer treatments may prefer an academically oriented applicant who has done well in oncology, whereas the hospital with a busy Emergency Room would prefer someone with practical skills who can work under pressure. A student might prefer a college where the courses are pitched at a level suitable to her ability; and the college might prefer a less able student because it has a mission, and the funding, to educate the less gifted, not just the cleverest. A research oriented university might prefer, and be preferred by, a scholar dedicated to research who sees teaching as a necessary chore, whereas a university whose income depends mostly on teaching may prefer a brilliant teacher to the star researcher.

The possibility that similar agents attracted to each other will actually match with each other has welfare and policy implications which go beyond the usual considerations of efficiency

[1] Interpreted literally, homophily means 'love of the same', and in this spirit I take it as a way to describe agents' tastes. In theoretical biology such preferences are sometimes labelled *homotypic;* for example, Alpern and Reyniers (1999). Fu et al, (2012) define homophily as the tendency to interact (not just to mate) with others of similar type. Buccara and Yariv (2013) and Currarini, Jackson and Pin (2009), in their anaysis of peer and friendship groups, also take homophily to be the association between similar people. I analyse a marriage market, and use the term homogamy when like actually matches with lilke.

[2] The literature on the determinants of personal attraction is enormous, so what follows is necessarily selective. On preferences for similarity (or not) of age see Lam (1988), Bergstrom and Lam (1991), Hayes (1995), Dalmia and Sicilian (2008); Eika, Mogstad, and Zafar (2019), Hitsch, Hortescu, and Ariely (2010), Ciscato, Galichon, and Gousse (2019), Sautman (2017). For similarity of culture and ethnic background see Rivera (2012‘), Currarini, Jackson, and Pini (2010), Boucher (2015), Boyd et al. (2003), Hitsch, Hortescu, and Ariely (2010), Fisman et al. (2008). For similarity of field-of-study amongst college graduates, see Bicakova and Jurajda (2016). For similarity of weight see Schafer and Keith (1990). For similarity of tastes for jointly consumed goods, see Clark and Kanbur (2004). For similarity of a range of personal traits see Buss and Barnes (1986) and for attraction to partners with similar affective disorders and other psychiatric traits see Mathews and Reus (2001). For similarity of religion see Carvalho (2016). For similarity of social rank see Goni (2018), Marcassa, Pouyet, and Trégouët (2019). On political similarity, see Huber and Malhotra (2017).

[3] See Baccara et al. (2010),

[4] For example, Besley and Ghatak (2005) analyse a model of organisational goals and workers' motivation in which principals would most prefer to employ (and are most preferred by) agents who share their ideals.

[5] Type of medical treatment is explicitly seen as a matching issue in Phelps (2000) and in Fabbri and Monfardini (2006)'s analysis of childbirth.

and the maximisation of total output. If attraction is based on similarity of tastes for music, for example, then there is no great social harm done if lovers of classical music match among themelves and not with afficianados of jazz. But if attraction is based on social position this raises the possibility of persistent social stratification. Attraction and sorting based on similarity of education provides a possible mechanism for the inheritance of social and economic advantage and the persistence of inequality, a point pursued by many authors (although mainly in a setting of *more is better).*[6] Racial preferences, whether for spouses, friends, adoptive children, employees, or in the assignment of teachers to schools, is clearly one avenue that might explain persistent racial inequality and segregation.

But although like may be attracted to like, this is no guarantee that like will actually be matched with like: homophily does not automatically imply homogamy. The route from preferences to outcomes depends on many factors, and indeed is the central question of matching theory. With fully transferable utility (TU), regardless of the distribution of types matching will display positive assortative matching (PAM) if the combined output or surplus of two matched agents is a supermodular function of their characteristics. In the case of homophilic preferences, supermodularity arises if the output q of a matched couple is a concave function of the difference in their types x and y;[7] for example $q = \alpha - |x - y|^2$. But even with PAM, whether couples are well matched will depend on the distribution of types. In the absence of supermodularity, matters are much more complicated, as the notion of submodularity generally makes little sense when like attracts like. For example, the function $\alpha - |x - y|^{0.5}$ is only submodular if $x - y$ does not change sign, in which case matching will display negative assortative matching (NAM). This is clearly very restrictive as it rules out pairs of type distributions (one for men, one for women) whose supports overlap.

Just as the pattern of matching is driven by factors many of which play no role in the standard model, so too agents' payoffs exhibit features not found when *more is better*. For example, with homophilic preferences, there is no reason for higher types (in the sense of a greater value of x or y) to receive a higher payoff. A high type female may be so different from all men that she can find a match only if she gets a small share of any joint output. This might suggest that the women most in demand (and thus with the highest payoffs) will be those with types close to the mass of male types. But as we shall see, this is not always the case: it is possible that women with types at or near the mean of the male type distribution have lower payoffs than all other women.

1.1 A simple example

To illustrate some of these issues, we suppose men and women are characterised only by height. If a man of height x is matched with a woman of height y, they produce output $q = \alpha - |x - y|^\beta$ where $\beta > 0$; he gets utility $u = q/2 - \pi$ and she gets $v = q/2 + \pi$, where π is a payment or

[6] For example Chiappori, Salanie, and Weiss. (2017), Ciscato, Galichon, Gousse (2019), Eika, Mogstad, and Zafar (2014), Fafchamps and Quisumbing (2002), Fernandez (2002), Kremer (1997), Olivetti, Paserman, and Salisbury (2018).

[7] In the language of the optimal transport literature, this is a case of convex costs. See Villani (2003, 2009) and Galichon (2016).

transfer, possibly negative, from him to her. Unmatched agents receive utility sufficiently low to ensure that everyone will be matched in equilibrium.

Figure 1 illustrates an important property of the relationship between types and output. If $\beta > 1$, q is a concave function of the absolute type difference $|x - y|$; we can also regard it as a concave function of the signed difference $x - y$, and hence as a supermodular function of x and y. If $\beta < 1$, q is concave function of $|x - y|$ but not of $x - y$, and is not, in general, a submodular function of x and y: $\partial q / \partial x$ is negative and decreasing in y for $y < x$, and positive and decreasing in y for $y > x$.

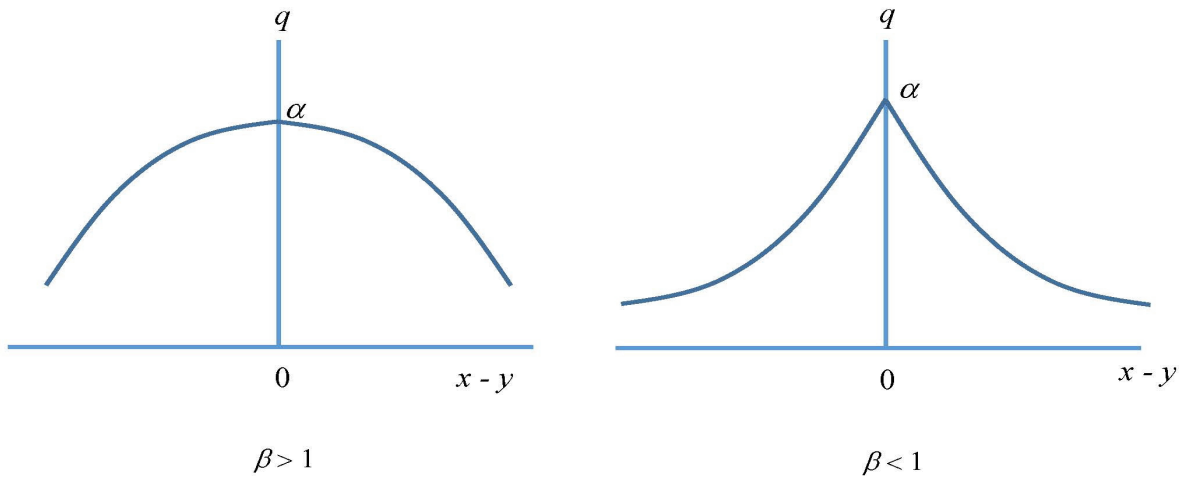


Figure 1: the relationship between q and $x - y$ when $q = \alpha - |x - y|^{\beta}$

We start with two women, Hillary and Melania, with heights (in centimeters) of 180 and 191 respectively, and two men, Donald and Bill, with heights 189 and 200; see Figure 2. We label this population P_1.

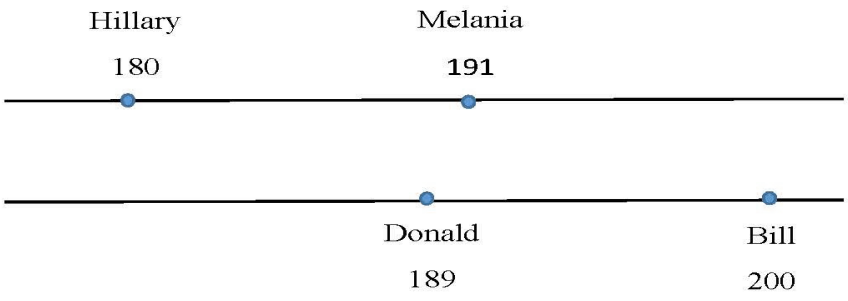


Figure 2: the population P_1

Consider first the case where $\beta = 2$. Total output is maximised by PAM, Hillary matching with Donald and Melania with Bill, so that (in obvious notation) $v_H + u_D = v_M + u_B = \alpha - 81$. The payoffs (and hence transfers) that sustain the PAM outcome are not unique. The requirement to prevent blocking pairs from forming implies $v_H + u_B \geq \alpha - 400$ and $v_M + u_D \geq \alpha - 4$; and if we narrow down the set of equilibria by exploiting the symmetry of the two distributions and impose $v_H = u_B$ and $v_M = u_D$ (so that neither gender is exogenously advantaged) then

$$\begin{aligned} \alpha/2 - 200 &\leq v_H = u_B \leq \alpha/2 - 79 \\ \alpha/2 - 2 &\leq v_M = u_D \leq \alpha/2 + 119 \end{aligned}$$

This shows very starkly the importance for Hillary's payoff of the position of her type not just in the distribution of women's types but also relative to the distribution of male types. She is the shorter person of the shorter sex and the alternative to an imperfect match with Donald is a bad match with Bill. But for Melania the alternative to Bill is a good match with Donald. Thus the taller woman gets the higher payoff. Similarly, Bill is the taller person of the taller sex, giving him as weak a position as Hillary's, and a lower payoff than Donald.

Suppose now $\beta = 1/2$. Total output is maximised by NAM, Hillary matching with Bill and Melania with Donald, so that $v_H + u_B = \alpha - 20^{1/2}$ and $v_M + u_D = \alpha - 2^{1/2}$. Again imposing gender equality, we immediately have

$$\begin{aligned} v_H &= u_B = \alpha/2 - 5^{1/2} \\ v_M &= u_D = \alpha/2 - 2^{-1/2} \end{aligned}$$

As $v_H + u_D = v_M + u_B > \alpha - 3$, these payoffs prevent the formation of blocking pairs and sustain NAM as an equilibrium. Again, Melania and Donald get the higher payoffs, despite the reversal of the matching pattern.

Note that with the types as given, PAM will generate greater total output than NAM if $\beta > \beta^*$, where $\beta^* \approx 0.569$ solves $20^\beta + 2^\beta = 9^\beta + 9^\beta$. But if we change the types distribution so that all women are shorter than all men (e.g. by swapping the heights of Melania and Donald) then NAM dominates PAM if and only if $\beta < 1$.

1.1.1 A symmetric population

Consider now a different population, P_2, of men Nicolas and Emmanuel, with heights 158 and 169, and women Brigitte and Carla, with heights 167 and 178. P_2 is a mirror image of P_1, reflected around 179, so any equilibrium of P_1 is an equilibrium of P_2, *mutatis mutandis*. If $\beta = 2$, we have PAM, Nicolas matched Brigitte, and Emmanuel with Carla, with $u_N + v_B = u_E + v_C = \alpha - 81$ and imposing gender equality:

$$\begin{aligned} \alpha/2 - 200 &\leq u_N = v_C \leq \alpha/2 - 79 \\ \alpha/2 - 2 &\leq u_E = v_B \leq \alpha/2 + 119 \end{aligned}$$

Similarly if $\beta = 1/2$ we have NAM and

$$\begin{aligned} u_N &= v_C = \alpha/2 - 5^{1/2} \\ u_E &= v_B = \alpha/2 - 2^{-1/2} \end{aligned}$$

For both values of β, it is now the shorter female and the taller male who get the higher payoffs.

1.1.2 A larger population

Merging P_1 and P_2 gives a population P_3, illustrated in Figure 3.

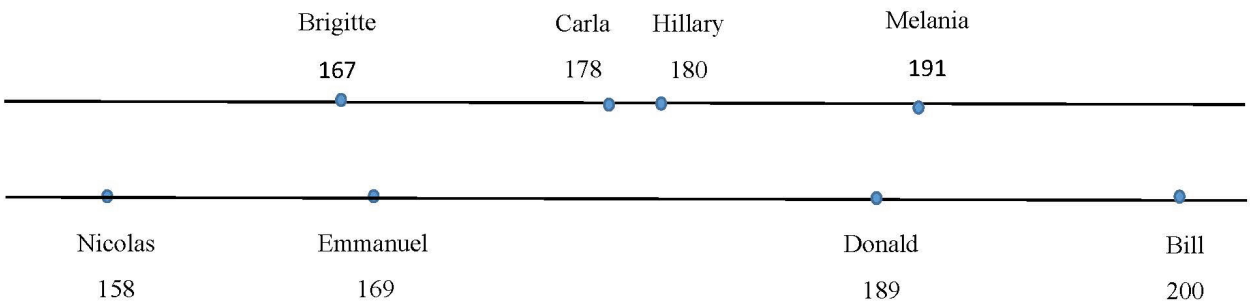


Figure 3: the population P_3

It can readily be computed that no output maximising matching of P_3 can have anyone from P_1 paired with anyone from P_2, whatever the technology; thus the equilibrium matching of P_3 combines those of P_1 and P_2. However, Nicolas's payoff, for example, need not be same as Bill's. We therefore narrow down the set of equilibrium payoffs by imposing a natural symmetry: that the common payoff of the shorter person of the shorter sex and the the taller person of the taller sex is the same in P_1 (Nicolas and Carla) as in P_2 (Hillary and Bill).[8] Thus if $\beta = 2$,

$$\alpha/2 - 200 \leq u_N = v_C = v_H = u_B \leq \alpha/2 - 79 \quad (1)$$

$$\alpha/2 - 2 \leq u_E = v_B = v_M = u_D \leq \alpha/2 + 119 \quad (2)$$

[8] This implies that the common payoff of the shorter person of the taller sex and the the taller person of the shorter sex is the same in P_1 (Brigitte and Emmanuel) as in P_2 (Donald and Melania),

where $u_N + v_B = u_E + v_C = v_H + u_D = v_M + u_B = \alpha - 81$. If $\beta = 1/2$

$$u_N = v_C = v_H = u_B = \alpha/2 - 5^{1/2} \quad (3)$$

$$u_E = v_B = v_M = u_D = \alpha/2 - 2^{-1/2} \quad (4)$$

In each case, the merging of P_1 and P_2 imposes no further constraints, as payoffs satisfying (1) and (2), or (3) and (4), also imply no-one in P_1 would prefer to match with anyone in P_2, and vice versa.

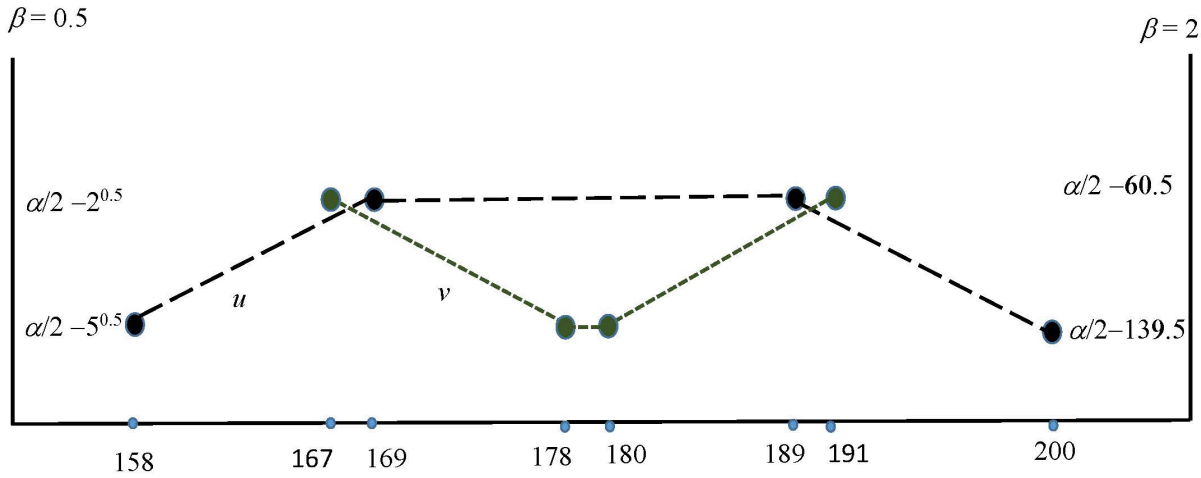


Figure 4: payoffs of the population P_3
left scale, $\beta = 0.5$; right scale, $\beta = 2$

Figure 4 shows payoffs as a function of type when $\beta = 0.5$ and $\beta = 2$; for the latter, we show the mid-point of the range of indeterminacy. In both cases, it illustrates several features that do not arise in the conventional model of *more is better*: (i) neither u nor v is monotonic; (ii) u and v move in opposite directions: u is increasing then decreasing whereas v is decreasing then increasing; (iii) if a man of height x is matched with a woman of type y, then u at x changes in the opposite way to v at y.

There is an additional phenomenon, which is at first counter-intuitive. Carla and Hillary have types in the middle of the male type distribution, so one might expect them to be in high demand; yet among the women they receive the lowest payoffs. As we shall see, this is not a peculiarity of this numerical example but quite generic and survives the extension to a more complete model. The explanation is that as women have the less dispersed distribution, there are more women than men with types at or around the common mean. In equilibrium this is resolved by a lower payoff for such women, and a higher payoff for Emmanuel and Donald. Similarly, the

men at the extremes of the male distribution, Nicolas and Bill, find it difficult to secure a close match, and have low payoffs, whereas Brigitte and Melania have heights that fill a particular niche, and have high payoffs. If we were to introduce an outside option, either for men or for women, that induced some married agents to leave the marriage market, these agents would be those with lower payoffs: the women in the middle of the type distribution and the men at the extremes.

As Figure 4 shows, the features described above are independent of the modularity or otherwise of the output function. But the technology matters in one important respect. For simplicity, we return to the population P_1. We now add one woman, Angela, and one man, Justin, both of height 185. When $\beta = 2$, PAM now means that Hillary is matched with Justin, Angela with Donald, and (as before) Melania with Bill. Angela and Justin, even though they are perfectly suited to each other, do not match in equilibrium. But if $\beta = 1/2$, it straightforward to show that Angela matches with Justin. This is quite general: when the output function is convex and a type t man matches with a type y woman, and at the same time a type t woman matches with a type x man (where $t \neq x, t \neq y$), the foursome could repartner, the t types together and the x type with the y type, and increase total output.[9] With convexity there will therefore be maximum matching of like with exactly like e.g. Angela and Justin. As I show, this has important implications for computing and characterising the overall matching, which is typically neither PAM nor NAM. For now, we note that in large populations it also means that at types where there are (for example) more men than women the men must be indifferent between choosing to match with women of the same type (so that we have maximum matching of like with exactly like) and women of some other type, possibly quite different (so that the excess men find a partner). Typically there will be many types either where men or women are relatively abundant. It is thus a generic feature of the convex case that for many agents (in the formal model a positive mass) their optimal choice is not unique; this is a significant departure from the standard case of *more is always better,* where supermodularity or submodularity give rise to PAM or NAM respectively and the equilibrium matching is characterised by a monotonic relationship between male and female types.

This result illustrates how tastes and technology interact to generate an efficient matching. Homophily naturally tends to bring similar agents together; but unless the type distributions are identical, there must be some imperfect matches. Concavity of the output function, via Jensen's inequality, smooths out the type gaps between matched agents, avoiding a mix of very good and very bad matches in favour of moderate gaps. With convexity, Jensen's inequality works the other way round: we have as many perfect matches as possible, with some of the remaining agents possibly very badly matched.

[9] If g is the output function then

$$g(0) + g(|x - y|) > g(|t - x|) + g(|t - y|)$$

which follows from $g' < 0$ (if t is either less or greater than both x and y) or from $g'' > 0$ (if t is between x and y).

1.2 Related theoretical literature

Although this paper looks at homophily in marriage markets, where matching is one-to-one, there is a considerable literature which look at homophily in other settings, particularly networks of people of similar beliefs, values, or attitudes; see, for example, Currarini, Jackson and Pin (2009), Bramoullé et al (2012), Baccara and Yariv (2013), the survey in Jackson (2014), and Currarini and Mengel (2016). The main aim of these papers is to explain and characterise networks as a result of homophilic preferences (which we also assume) and bias in meeting opportunities (which we do not). Recent work by Kets and Sandroni (2019) has sought to explain the tendency to associate with similar people not as a result of innate homophilic preferences, but as a way to reduce uncertainty about others' actions: agents find it easier to empathise with similar types and hence to predict their behaviour. Intriguingly, agents may form homophilic connections even if this is at the cost of not satisfying non-homophilic preferences.

In addition to the many applied papers on homophily and matching in marriage markets, both in economics and other subjects, there is an important theoretical literature with origins largely outside Economics. The problem of 'optimal transportation' was posed by Monge (1781) in the context of minimising the costs of transferring iron ore from mines to factories: if we seek a one-to-one matching of mines and factories and transportation is costly, what is the optimal matching? The parallel with seeking a matching of men and women that minimises the differences between them is obvious. Kantorovich formalised this as a programming problem, and developed and applied to it the powerful tools of duality theory; in particular, he showed how prices can support the optimal outcome as a decentralised equilibrium (for example, see Kantorovich (2006), a translation of a 1944 paper). Many applications and further topics in the mathematics of optimal transportation can be found in Villani (2003) and Villani (2009). The monographs by Galichon (2016) and Chiappori (2017) illustrate how fruitful this approach is in analysing a wide range of economic situations, and in particular to showing how optimal transport methods complement and extend the pioneering work of Becker (1973) in analysing matching and sorting.

Most economic applications have not been in a framework of minimising the cost of differences between matched agents but rather in one where *more is always better*. In many applications, this may be the more suitable assumption e.g. where two inputs both have positive marginal products, However, that is not always the case, as shown by the range of applications where *like attracts like* is more appropriate. There has been a small number of theoretical papers that assume the output of two matched agents is decreasing in some measure of the difference betwen them. Clark (2007) has a set-up similar to that considered here, but assumes non-transferable utility and provides no analysis of payoffs. Klumpp (2009) models spatially differentiated agents but allows trade only in one direction (i.e. buyers and sellers are on opposite sides of a one-way street, but trade only occurs if the seller is to the left of the buyer). Hofmann and Qari (2010) analyse a model of search and matching, but they only consider non-transferable utility, and agents in their model have types uniformly distributed on a circle. The model presented in this paper has types distributed on the real line, and allows trade in both directions (e.g. tall men may match with short women, and tall women with short men). Furthermore, in assuming transferable utility we are able to take full advantage of the duality results of Kantorovich, Villani,

and Galichon; in particular we show how payoff functions can display unexpected properties.

1.3 Plan of the paper

The rest of the paper is organised as follows. In the next section, I set up a formal model of frictionless matching with transferable utility, with a restriction on the surplus generating technology that embodies homophily. I derive a number of properties of the payoff functions that are independent of any further assumptions about the technology. In Section 3, I analyse matching and payoffs when output is a decreasing and concave function of the type difference between matched agents, equivalent to supermodularity. Section 4 looks at the case when the output function is decreasing but convex. This case is not equivalent to submodularity, and poses a range of analytical challenges. Nevertheless, the matching pattern is shown to obey certain principles, in particular a Two Type Property, whereby some agents can achieve their equilibrium payoff by choosing a partner of their own type or of some other type. Section 5 looks at the payoff functions in more detail; in so far as they are wavelike, I analyse what determines their amplitide and frequency. Section 6 extends the model to allow for unmatched agents to have an outside option. We are then able to show who remains single and which are married. Section 7 concludes. Throughout, I illustrate the analysis with variations on a small number of simple numerical examples. Proofs are kept to an Appendix.

2 The Model

Agents are characterised by gender and type. An agent's type is a real number in the interval $T = [\underline{t}, \overline{t}]$. Integrable functions $\phi : T \to R$ and $\gamma : T \to R$ give the density of types amongst men and women respectively, with interval supports T_ϕ and T_γ and distribution functions Φ and Γ, where $T_\phi \cup T_\gamma = T$. There is an equal mass Ω of men and women, so that $\Phi(\overline{t}) = \Gamma(\overline{t}) = \Omega$. If a man of type x is matched with a woman of type y they produce an output $f(x, y)$, where $f : T \times T \to R$, and get utility $f(x, y)/2 - \pi$ and $f(x, y)/2 + \pi$ respectively where π is a payment, possibly negative, from him to her; an unmatched agent gets zero utility. To capture the idea that like attracts like, I assume

Assumption 1: $f(x, y) = g(|x - y|)$ for some twice differentiable function g with domain $[0, \overline{t} - \underline{t}]$ such that

$$0 < g(|x - y|) \text{ for } |x - y| \in [0, \overline{t} - \underline{t}] \tag{5}$$

$$g'(|x - y|) < 0 \text{ if } x \neq y. \tag{6}$$

Thus $f_x(x, y) = -f_y(x, y)$ if either $x \neq y$ or $x = y$ and $g'(0) = 0$. Note that, as in the special case illustrated in Figure 1, if g is concave then f is supermodular, but if g is convex, f is not generally submodular. However, if the supports of ϕ and γ do not overlap, $x - y$ does not change sign, so if g is convex, $f_{xy} = -g''(|x - y|) < 0$ for all $(x, y) \in T_\phi \times T_\gamma$; f is then *effectively submodular*. For simplicity, from now on we write $g(|x - y|)$ as $g(x - y)$, it being understood that g admits only a non-negative argument.

A *matching* describes what types of men are matched with what types of women, and is a measure on T^2 with marginals ϕ and γ.[10] For example, given a matching θ, $\theta(A \times B)$ denotes the mass of matched pairs such that the man's type is in A and the woman's is in B. Similarly, if D is the 45^o line in $T \times T$, $\theta(D)$ measures the extent of homogamy (like matched with exactly like). Then f is a measurable function, and the total output from a matching θ is $Q = \int_{T \times T} f d\theta$.

Partitions of T In determining the matching pattern and payoff functions, a key role is played by the intersections of the density functions ϕ and γ, and of the distribution functions Φ and Γ. To rule out trivial cases where the density functions differ at a set of points of measure zero, and pathological cases involving an infinite number of intersections, I assume:

Assumption 2: If $\phi \neq \gamma$ then for some finite number $n > 1$ the interval T has a partition $(\underline{t}, t_1, ..., t_{n-1}, \overline{t})$,with n subintervals $T_i = [t_{i-1}, t_i]$ where $t_0 = \underline{t}$ and $t_n = \overline{t}$, such that for $i = 1, ..., n$

either $\phi(t) > \gamma(t)$ for all $t \in int(T_i)$, (then T_i is a ϕ-*interval*)

or $\phi(t) < \gamma(t)$ for all $t \in int(T_i)$, (then T_i is a γ-*interval*)

and no two adjacent intervals are of the same type.

Note that, given ϕ and γ, if two partitions both satisfy Assumption 1, they must be the same.[11] The intervals of this unique partition have a natural ordering, expressed as $T_i < T_j$ for $i < j$. Assumption 2 has immediate implications for the intersections of Φ and Γ :

Lemma 1 If $\phi \neq \gamma$, then $\Phi \neq \Gamma$ and the interval T has a unique partition $(\underline{t}, s_1, ..., s_{m-1}, \overline{t})$, with m subintervals $S_i = [s_{i-1}, s_i]$ where $s_0 = \underline{t}$ and $s_m = \overline{t}$ such that

(i) $\Phi(s_i) = \Gamma(s_i)$ for $i = 0, 1, ..., m$;

(ii) either $\Phi(t) > \Gamma(t)$ for all $t \in int(S_i$ (then S_i is a Φ-*interval*)

or $\Gamma(t) > \Phi(t)$ for all $t \in int(S_i)$ (then S_i is a Γ-*interval*)

(iii) $m < n$.

2.1 Equilibrium with TU

The powerful duality results that follow from the framework above have been widely exploited in the matching literature. For a recent exposition, see Galichon (2016), in particular Proposition 2.3 and Theorem 7.6.[12] The key feature is that a matching θ that maximises total output Q can be sustained as a competitive equilibrium; i.e. there exist payoff functions u and v, with domains T_ϕ and T_γ respectively, such that

$$u(x) + v(y) \geq f(x, y) \text{ for all } (x, y) \in T_\phi \times T_\gamma \quad (7)$$

$$u(x) + v(y) = f(x, y) \text{ if and only if } (x, y) \in supp(\theta) \quad (8)$$

[10] If θ is a matching then $\theta(A \times T) = \int_A \phi(x)dx$ and $\theta(T \times B) = \int_B \gamma(y)dy$. Note that this assumes all agents are matched. Section 6 extends the model to allow for unmatched agents.

[11] If not, there must exist some t such that two partitions assign a different value to $\gamma(t) - \phi(t)$, a contradiction.

[12] Our use of Galichon's Theorem 7.6 relies on the boundedness condition (6).

where $supp(\theta)$ denotes the support of θ. Thus

$$u(x) = \max_y f(x,y) - v(y) \quad (9)$$

$$v(y) = \max_x f(x,y) - u(x) \quad (10)$$

and the choices made by types x and y when they take as given the functions v and u respectively in turn give the matching θ. This allows us first to consider the matching that maximises total output, and then to analyse the payoff functions that sustain this matching as an equilibrium.

2.2 The wavelike or periodic nature of the payoff functions.

We can identify some features of u and v that are independent of any further assumptions about f and g and thus hold for any matching pattern.

Proposition 1 *(i) u and v are both continuous.*
(ii) For any t in $T_\phi \cap T_\gamma$ if u is increasing (resp. decreasing) at t, then v is decreasing (increasing) at t.
(iii) if $(x,y) \in supp(\theta)$ and $x \neq y$, then $u'(x) = -v'(y)$.

These three points put a particular structure on the payoff functions. Not surprisingly, *like attracts like* produces important differences from *more is better*. Higher types do not necessarily receive a larger payoff. When like attracts like, if a man of type x is taller than his partner of type y, then a marginally shorter man, with type $x' < x$, could match with the y type and generate a higher joint output. This creates a rent for x' so that $u(x') > u(x)$; i.e. u is decreasing at x. By the same reasoning, v is increasing at y.

Taking all three points together, we can see that u and v behave in an opposite way to each other: if x types match with y types and x and y are both in $T_\phi \cap T_\gamma$ then if u is increasing at x it is also increasing at y, and v is decreasing at both x and y. Then if u has a local minimum or maximum at t, v has a local maximum or minimum respectively at t.

Two special cases are of interest. Firstly, if $\phi = \gamma$, then unless almost all agents are perfectly matched aggregate output is less than $\Omega g(0)$. To induce each agent to choose a same-type partner requires that u and v do not change too much. More precisely, suppose $|v(t) - v(t')| \leq g(0) - g(t-t')$. Then a man of type t cannot do better by choosing a woman of type t over one of type t'; nor can a man of type t' do better by choosing a woman of type t' over one of type t. But if $|v(t) - v(t')| > g(0) - g(t-t')$, then either a type t man prefers a type t' woman or a type t' man prefers a type t woman. Thus if $\phi = \gamma$ then

$$\left|v(t) - v(t')\right| \leq g(0) - g(t-t') \text{ for any } (t,t') \in T \times T \quad (11)$$

A similar argument applies to women's choices; but simpler here is to note that if all agents are perfectly matched then $u + v = g(0)$, so (11) implies

$$\left|u(t) - u(t')\right| \leq g(0) - g(t-t') \text{ for any } (t,t') \in T \times T \quad (12)$$

These conditions are satisfied if (but generally not only if) u and v are constant.

As a second special case, suppose all men are taller than all women. Then u must be strictly decreasing over its domain T_ϕ; otherwise there would be some men who, being taller and more expensive, would never find a partner. Similarly, v must be increasing over T_γ. If all men are shorter than all women, then u is increasing and v decreasing.

3 Concave g (supermodular f)

We now assume that g is strictly concave, or equivalently that f is supermodular. As is well known, this implies that an equilibrium matching will display PAM (Lorentz (1953), Becker (1973)). We represent this by the bijection $\mu : T_\phi \to T_\gamma$ defined by $\Phi(t) = \Gamma(\mu(t))$. Then

$$\begin{aligned} \mu(x) &= \arg\max_y [f(x,y) - v(y)] \\ \mu^{-1}(y) &= \arg\max_x [f(x,y) - u(x)] \end{aligned}$$

If the two distributions are the same, then the maximum possible total output is $\Omega g(0)$, which cannot be achieved if a positive measure of couples are imperfectly matched. And since it is not possible to match all (or even almost all) agents perfectly if $\Phi \neq \Gamma$, then $\mu(t) = t$ for almost all $t \in T$ (i.e. homophily implies homogamy) if and only if $\Phi = \Gamma$. If $\Phi \neq \Gamma$, then only at points where $\Phi(t) = \Gamma(t)$ does like match with exactly like. Nevertheless, with g concave, there is a strong tendency to homogamy: concavity places a small cost on small departures from perfect matching, and an increasingly high cost on further departures, and the type differences of matched couples are evened out through PAM.

3.1 The structure of payoffs

3.1.1 The type distributions are the same

As noted above, if $\phi = \gamma$, almost everyone is perfectly matched. If g is concave, the conditions (11) and (12), limiting the rate of change in u and v, can only be satisfied if, for any $(t, t') \in T \times T$,

$$\begin{aligned} \left|v(t) - v(t')\right| &\leq \left|g'(0) \times (t - t')\right| \\ \left|u(t) - u(t')\right| &\leq \left|g'(0) \times (t - t')\right| \end{aligned}$$

In particular, if $g'(0) = 0$ then both u and v are constant.

3.1.2 The type distributions are not the same.

If $\phi \neq \gamma$ then from Lemma 1 $\Phi \neq \Gamma$. The payoff functions then have a structure that is determined by the intersections of Φ and Γ. To see this, suppose that $\Phi(t) > \Gamma(t)$ for some type t. Then $\mu(t) > t$ and $t > \mu^{-1}(t)$, so if either partner is of type t then the man is of a lower type than the

woman. Applying the analysis of Section 2.2, u is increasing and v is decreasing both at t and at $\mu(t)$; similarly, if $\Phi(t) < \Gamma(t)$, u is decreasing and v is increasing both at t and at $\mu(t)$

Thus u and v have an alternating or wavelike pattern, governed by the behaviour of $\Phi - \Gamma$, which equals zero at the $m+1$ points $\underline{t}, s_1, ..., s_{m-1}, \bar{t}$ and changes sign $m-1$ times. With PAM, $\mu(s_i) = s_i$ at each s_i, and if t is in the interior of S_i then so are both $\mu^{-1}(t)$ and $\mu(t)$. These intervals define m self-contained strata or "mini-economies": agents with types in S_i match only with other agents with types in S_i. In Φ-intervals u is increasing and v is decreasing; and the opposite in Γ-intervals. Thus the wavelike stucture of the payoff functions follows if $m > 1$.

At the boundaries between intervals, $f_x(s_i, s_i)$ and $f_y(s_i, s_i)$ are not defined unless $g'(0) = 0$, so although the payoff functions are continous, they may have kinks at these points. But as there is a finite number of boundary points, we can still write the equilibrium payoff functions in the usual way:

$$\begin{aligned} u(x) &= \int_{x_0}^{x} f_x(t, \mu(t))dt + u_0 \\ v(y) &= \int_{y_0}^{y} f_y(\mu^{-1}(t), t)dt + v_0 \end{aligned}$$

where we take u_0 and v_0 to be exogenously given such that $u_0 + v_0 = f(x_0, \mu(x_0))$ for some $x_0 \in T_\phi$.

Note that an agent's payoff may bear only a very weak relationship to the quality of the match they are in. Indeed, at a point, say s_i, where $\Phi = \Gamma$ and u is at a local minimum, a perfectly matched man, with utility $u(s_i)$, will be surrounded (in type space) by imperfectly matched men with a higher payoff; and since v is at a local maximum at that point, a perfectly matched woman will be surrounded by women with a lower payoff. At the next intersection of Φ and Γ, s_{i+1}, u is at a local maximum and a perfectly matched man gets utility $u(s_{i+1})$,so that $u(s_{i+1}) > u(s_i)$. With a wavelike pattern of payoffs, agents who are perfectly matched but of different types typically do not get the same payoff as each other.

The analysis above allows us to draw some general conclusions about which types are better off than others. We have shown that u is increasing and v decreasing in Φ-intervals, and the opposite in Γ-intervals. If men tend to be shorter than women, then Φ-intervals predominate (in the sense of covering more of the type space T than Φ-intervals), and on the whole it is advantageous to be a taller man and a shorter woman. On the other hand, if the distribution of male types is the more dispersed then Φ-intervals predominate at the lower end of the type space and Γ-intervals at the higher end, so that u tends to be initially rising and then falling, while v does the opposite; it is then better to be a man in the midde of the male distribution, whereas the better off women are those at the extremes of the female distribution.

3.2 Examples using $f(x,y) = \alpha - |x-y|^\beta$ with $\beta > 1$

3.2.1 Example (i): distributions that differ only in mean

We take x and y to be uniformly distributed, with $\phi(x) = 1$ for $x \in [0,1]$ and $\gamma(y) = 1$ for $y \in [1,2]$. Thus $n = 2$, $m = 1$ and $[0,2]$ is a Φ-interval. With these type distributions $\mu(x) = 1 + x$, so that $f(x, \mu(x)) = \alpha - 1$. Equilibrium payoffs are given by

$$\begin{aligned} u(x) &= u_0 + \beta x \\ v(y) &= v_1 + \beta(1-y) \end{aligned}$$

where $u_0 + v_1 = \alpha - 1$. In this example, it is an advantage to be close to the mass of agents on the opposite side.

3.2.2 Example (ii): distributions with the same mean but different dispersion

Now let $\phi(x) = 1$ for $x \in [1,2]$ and $\gamma(y) = 1/3$ for $y \in [0,3]$. Thus $n = 3$ and $m = 2$; $[-0, 1.5]$ is a Γ-interval, and $[1.5, 3]$ is a $\Phi-$interval. Now $\mu(x) = 3(x-1)$; for $x < 1.5$ women are shorter than their partners, and for $x > 1.5$ they are taller. Payoff functions are given by

$$\begin{aligned} u(x) &= u_{1.5} + |3 - 2x|^\beta / 2 \\ v(y) &= v_{1.5} - 3^{1-\beta} |3 - 2y|^\beta / 2 \end{aligned}$$

where $u_{1.5} + v_{1.5} = \alpha$. Figure 5 illustrates with $\beta = 2$.

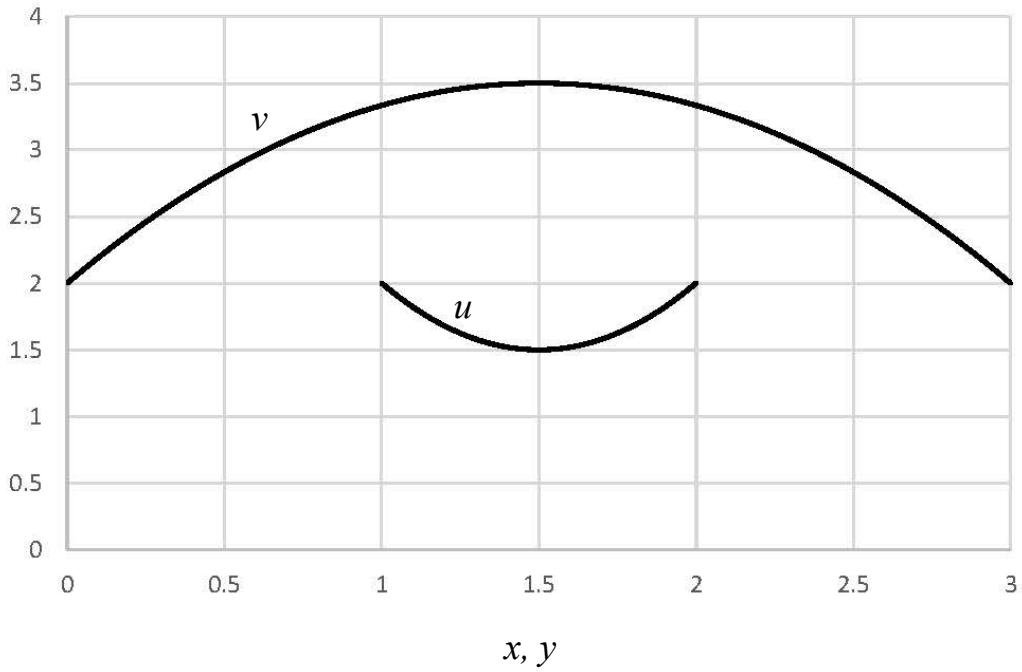


Figure 5: payoff functions in example (ii) when $\beta = 2$

In this example, men with types close to 1.5, the mean of both distributions, are disadvantaged, whereas those with types near 1 or 2 benefit from being close to the more extreme female types, for whom a close match is difficult.

3.2.3 Example (iii) : the wavelike pattern of payoffs

We let $\phi(x) = 2$ for $0 \leq x \leq 6$ and

$$\gamma(y) = \begin{cases} 3 & 0 \leq y < 1 \\ 1 & 1 \leq y < 3 \\ 3 & 3 \leq y < 5 \\ 1 & 5 \leq y \leq 6 \end{cases}$$

Thus $n = 4$ and $m = 3; \Omega = 12$; $\Phi(t) = \Gamma(t)$ at $t = 0, 2, 4$, and 6; and $[0, 2], [2, 4]$, and $[4, 6]$ are Γ, Φ, and $\Gamma-$intervals respectively. The PAM matching μ is given by

$$\mu(x) = \begin{cases} 2x/3 & 0 \leq x \leq 1.5 \\ 2x - 2 & 1.5 \leq x \leq 2.5 \\ 1\frac{1}{3} + 2x/3 & 2.5 \leq x \leq 5.5 \\ 2x - 6 & 5.5 \leq x \leq 6 \end{cases}$$

The payoff functions that sustain this matching are

$$u(x) = \begin{cases} u_0 - 3^{1-\beta}x^\beta & 0 \leq x \leq 1.5 \\ u_0 - 2^{2-\beta} + |2 - x|^\beta & 1.5 \leq x \leq 2.5 \\ u_0 - 3^{1-\beta}|4 - x|^\beta & 2.5 \leq x \leq 5.5 \\ u_0 - 2^{2-\beta} + |6 - x|^\beta & 5.5 \leq x \leq 6 \end{cases}$$

$$v(y) = \begin{cases} v_0 + 2^{1-\beta}y^\beta & 0 \leq y \leq 1 \\ v_0 + 2^{2-\beta} - 2^{1-\beta}|y - 2|^\beta & 1 \leq y \leq 3 \\ v_0 + 2^{1-\beta}|4 - y|^\beta & 3 \leq y \leq 5 \\ v_0 + 2^{2-\beta} - 2^{1-\beta}|6 - y|^\beta & 5 \leq y \leq 6 \end{cases}$$

where $u_0 + v_0 = \alpha$. Figure 6 illustrates with $\beta = 2$.

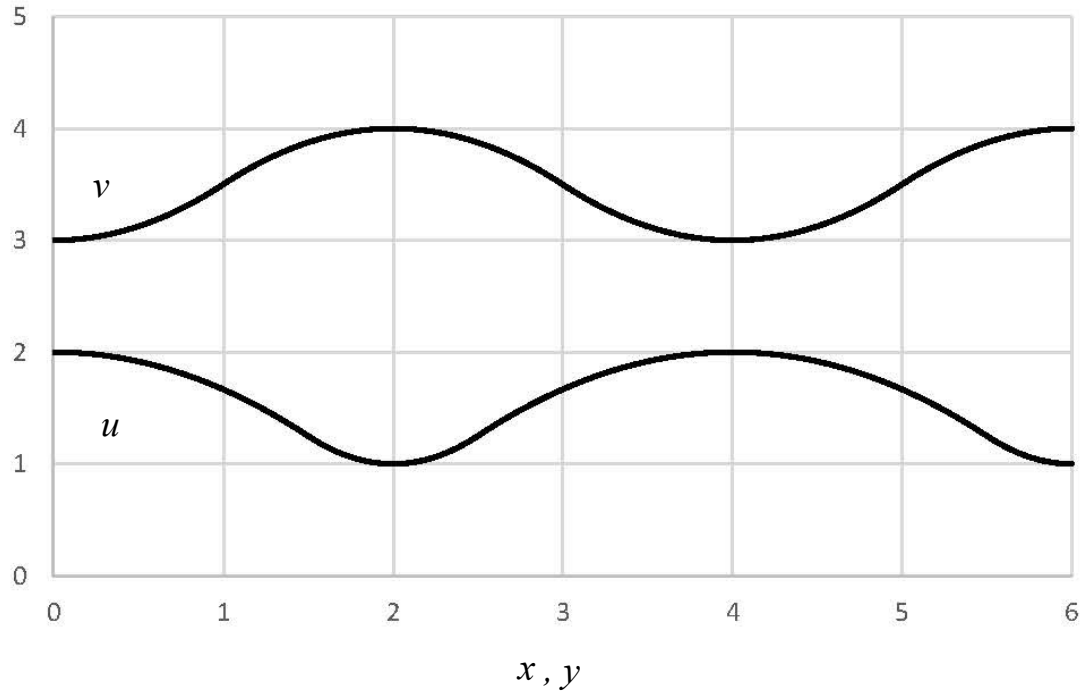


Figure 6: payoffs in example (iii) when $\beta = 2$

4 Convex g (f not supermodular)

We now assume that g is convex, reflecting a diminishing marginal loss of surplus as the type difference $|x - y|$ increases. This introduces a range of complications and we need new ways to find the matching that maximises output. The convexity of g places a high initial cost on small departures from perfect matching and a low cost on further departures; the tendency to even out the type differences of matched couples that arises from homophilic preferences is thereby greatly reduced, but not entirely eliminated. Although the equibrium matching now typically exhibits neither PAM nor NAM, it nevertheless obeys some simple principles which we derive and illustrate. We begin with two special instances which turn out to be important building blocks in the more general case.

4.1 The type distributions are the same

As in the concave case, when $\phi = \gamma$ we have PAM with almost everyone perfectly matched: any other matching implies that aggregate output is less than $\Omega g(0)$. It is still the case that perfect matching is sustained by payoff functions that do not change too much, and in particular when u and v are both constant. However, consider the limits of the variation in the payoff functions permitted by conditions (11) and (12): if (12) holds with equality for a pair (t, t') and $v(t) > v(t')$, then a man of type t can achieve his equilibrium payoff by choosing a woman of his own type or of type t'; i.e. his maximand has two peaks, of exactly equal height, as does the maximand of a type t' woman, since $u + v = g(0)$. This feature of the equilibrium payoffs when g is convex plays a central role in the more general case, as it allows us to construct payoff functions where agents

have two maximising options.

4.2 No overlap between the type distributions

Suppose no man is taller and no woman is shorter than t'; i.e. $x \leq y$ for $(x, y) \in T_\phi \times T_\gamma$, with equality only if $x = y = t'$. Then $f_{xy}(x, y)$ always exists and is negative for $(x, y) \in T_\phi \times T_\gamma$.[13] Thus f is effectively submodular, and we have NAM. We represent this by the bijection $\lambda : T_\phi \to T_\gamma$ defined by $\Phi(t) + \Gamma(\lambda(t)) = \Omega$ so that $\lambda'(t) = -\phi(t)/\gamma(\lambda(t))$. Then the payoff functions satisfy

$$\lambda(x) = \arg\max_{y \in T_\gamma} [f(x, y) - v(y)] \tag{13}$$

$$\lambda^{-1}(y) = \arg\max_{x \in T_\phi} [f(x, y) - u(x)] \tag{14}$$

so that in equilibrium:[14]

$$\begin{aligned} u'(x) &= f_x(x, \lambda(x)) \text{ for } \underline{t} \leq x \leq t' \\ v'(y) &= f_y(\lambda^{-1}(y), y) \text{ for } t' \leq y \leq \overline{t} \end{aligned}$$

If men are shorter than their partners, f_x is positive and f_y negative, so u is increasing and v decreasing; if women are the shorter sex, f_x is negative and f_y positive, so u is decreasing and v increasing. Furthermore

$$\begin{aligned} u''(x) &= f_{xx}(x, \lambda(x)) \left(1 - \lambda'(x)\right) \text{ for } \underline{t} \leq x \leq t' \\ v''(y) &= f_{yy}(\lambda^{-1}(y), y) \left(1 - 1/\lambda'(\lambda^{-1}(y))\right) \text{ for } t' \leq y \leq \overline{t} \end{aligned}$$

where we have used the fact that if $x \neq y$ then $f_{xx} = -f_{xy} = f_{yy}$. As $\lambda' < 0$, both u and v are convex; indeed $u''(x) > f_{xx}(x, \lambda(x))$ and $v''(y) > f_{yy}(\lambda^{-1}(y), y)$, which are effectively the second order conditions for the maximisations in (13) and (14) respectively.

Thus despite the reversal of the matching pattern, the payoff functions are similar to the PAM case with $m = 1$. This result complements the insight of Eeckhout and Kircher (2011) that the matching pattern (PAM or NAM) cannot be identified just from the slope of the payoff functions. In their set-up, more x or more y is always better; here, with x always less than y, more x or less y is always better, so regardless of the modularity or not of f, u is increasing and v is decreasing.

4.2.1 Example (i) revisited

We reconsider the case where male and female types are uniformly distributed with unit density on $[0, 1]$ and $[1, 2]$, so $\lambda(x) = 2 - x$. As before, $f(x, y) = \alpha - |x - y|^\beta$ but with $\beta < 1$ payoffs are

[13] Note that $f_{xy}(t', t') = g''(0)$, being the limit of $f_{xy}(x, t')$ as x tends to t' from below.

[14] Note that although f is not concave in x or in y,the first order conditions for x's choice of y and y's choice of x can only be satisfied if $\lambda^{-1}(y) = x$ and $\lambda(x) = y$ respectively. The second order conditions evaluated at these points reduce to $g''/\lambda' < 0$ and $g'' \times \lambda' < 0$, which are satisfied due to the convexity of g and NAM, so that $\lambda(x)$ and $\lambda^{-1}(y)$ are indeed the maximising choices for types x and y respectively.

given by

$$u(x) = u_1 - 2^{\beta-1}(1-x)^{\beta} \qquad\qquad v(y) = v_1 - 2^{\beta-1}(y-1)^{\beta}$$

where $u_1 + v_1 = \alpha$.

4.3 The matching pattern when the type distributions overlap

4.3.1 Maximum possible matching of like with like

We show now that when g is convex and the type distributions overlap, there is maximum possible overall matching of like with like. There is maximum matching of like with like *for a particular type* t if the density of type t men who are matched with type t women is $\kappa(t) = \min\{\phi(t), \gamma(t)\}$. Then $\int_T \kappa(t)dt$ is an upper bound on how much of the population can be in a perfect match; i.e. $\theta(D) \leq \int_T \kappa(t)dt$ for any matching θ. But suppose $\theta(D) < \int_T \kappa(t)dt$, and consider a man of type t matched with a woman of type y and a woman of type t matched with a man of type x, where $y \neq t$ and $x \neq t$. Then this foursome can repartner, the t types together, and the x type with the y type, and increase total output. That is to say

$$g(0) + g(|x-y|) > g(|t-x|) + g(|t-y|) \tag{15}$$

which follows from $g' < 0$ (if t is either less or greater than both x and y) or from $g'' > 0$ (if t is between x and y). If $\theta(D) < \int_T \kappa(t)dt$, then the repartnering described above can occur on a sufficient scale to produce a measurable increase in overall output. We therefore have the principle of Maximum Matching of Like with Like:

> **Proposition 2** If g is convex and θ is an output maximising matching then $\theta(D) = \int_T \kappa(t)dt$.

4.3.2 Imperfect matching of the remaining subpopulation

Once we have matched like with like to the maximum possible extent, we are left with a subpopulation with types given by densities ϕ^* and γ^*, where $\phi^*(t) = \max(\phi(t) - \gamma(t), 0)$ and $\gamma^*(t) = \max(\gamma(t) - \phi(t), 0)$. Thus the support of ϕ^* (resp. γ^*) is the union of the ϕ intervals (resp. $\gamma-$ intervals) of T. Suppose $i < j$, where T_i is a $\phi-$ interval and T_j is a $\gamma-$ interval, and that a positive measure of men with types in T_i match with women with types in T_j. Then since all the men are shorter than all the women, this "submatching" must display NAM; this is just a simple extension of the no overlap analyis of Section 4.2. If T_i is a $\gamma-$ interval and T_j an $\phi-$ interval then all the women are shorter than all the men but again we have NAM.

The matching of the subpopulation with densities ϕ^* and γ^* thus consists of a number of such submatchings, each displaying NAM. The NAM submatching between agents with types T_i and those with types in T_j, where $i < j$, is denoted by θ_{ij}. By construction, $j - i$ is odd; if $j - i = 1$, then θ_{ij} is an *adjacent submatching*. If θ_{ij} matches a positive measure of such agents then we say that θ_{ij} has positive measure; the number of NAM submatchings with positive measure is denoted by N .

4.3.3 Almost perfect matching

Although there are no further perfect matches to be had once we have maximum matching of like with like, there are still some almost perfect matches available to couples whose types are close to each other but on opposite sides of the boundary between a $\phi-$interval and an adjacent $\gamma-$interval. Thus:

> **Proposition 3**. If q is convex, every possible adjacent submatching has positive measure.

As ϕ and $\gamma-$intervals alternate, there are $n-1$ possible adjacent submatchings. Although each exhibits NAM, if $n > 2$ there will be PAM *between* them; e.g. for $n = 3$ with $\phi-$intervals T_1 and T_3 and a $\gamma-$interval T_2, very short men match with moderately short women (θ_{12}), and moderately tall women match with very tall men (θ_{23}).

4.3.4 Forbidden Foursomes

Given the n intervals of the partition $(\underline{t}, t_1, ..., t_{n-1}, \bar{t})$, if there were a submatching of positive measure between each $\phi-$ interval and each $\gamma-$ interval, then we would have $N = n^2/4$ if n is even or $N = (n^2-1)/4$ if n is odd. However, there are some combinations of submatchings that we can rule out as inefficient. These fall into two categories, illustrated in Figure 7, where $T_i < T_j < T_k < T_l$.

Category A. Suppose two intervals of one type lie beween two of the other; for example, T_i and T_l are $\gamma-$ intervals and T_j and T_k are $\phi-$ intervals. Then if the θ_{jl} and the θ_{ik} sub-matchings both have positive measure there is an obvious inefficiency, as we could reassign agents from those sub-matchings to the θ_{ij} and θ_{kl} submatchings, reducing the distance between partners' types and increasing total output.

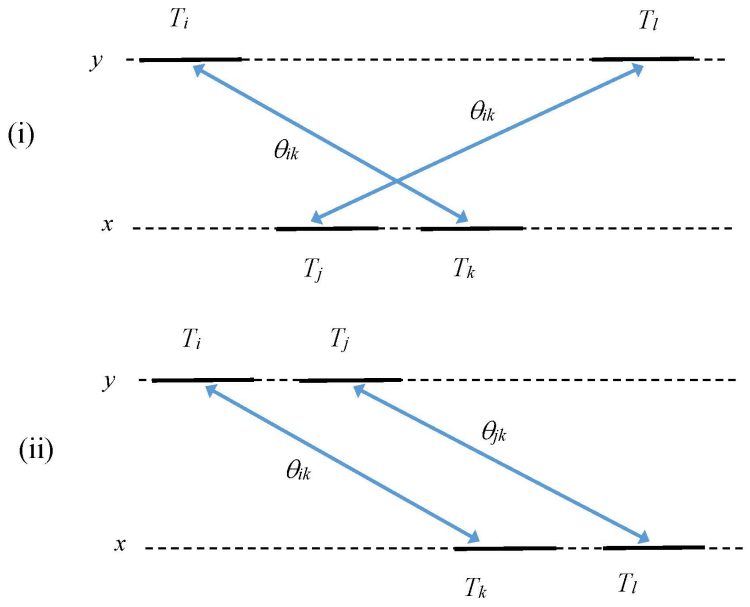


Figure 7: (i) Category A inefficiency; (ii) Category B inefficiency

Category B. Suppose the two intervals of one type contain agents that are shorter than agents in the two intervals of the other type; for example, T_i and T_j are $\gamma-$ intervals and T_k and T_l are $\phi-$ intervals. Then all women are shorter than all men and overall we must have NAM, so the θ_{ik} and the θ_{jl} submatchings cannot both have positive measure.

The following is immediate:

Proposition 4. If any submatching θ_{ij} has positive measure, the subsets of T_i and T_j that it matches are themselves intervals.

With these results we can now put further structure on the equilibrium matching pattern, which in turn has implications for the structure of payoffs.

4.3.5 The Watershed Principle

Of the imperfectly matched men with types in a $\phi-$interval T_i, some will match with women with lower types (i.e. in $\gamma-$intervals below T_i) and some with women with higher types (in $\gamma-$intervals above T_i). But if a man with a type in T_i matches with a woman with a lower type than his (e.g. in T_{i-1}) yet has a higher type than a man, also with a type in T_i, who matches with a woman with a higher type (e.g. in T_{i+1}), then we have a Category A inefficiency (with T_i the union of the two middle intervals). This implies that each $\phi-$interval T_i contains a critical type (or *watershed)* x_i^*, such that imperfectly matched men with types in T_i below x_i^* match only with women with types below T_i and imperfectly matched men with types in T_i above x_i^* match only with women with types above T_i. Similarly, each $\gamma-$interval T_j contains a critical type y_j^*, such that imperfectly matched women with types in T_j below y_j^* match only with men with types below T_j and imperfectly matched women with types in T_j above y_j^* match only with men with types above T_j.

Clearly the watersheds in T_1 and T_n must be $\underline{t}$ and $\bar{t}$ respectively, but since all adjacent NAM submatchings have positive measure, $t_{i-1} < x_i^* < t_i$ for $1 < i < n$ if T_i is a $\phi-$interval and $t_{j-1} < y_j^* < t_j$ for $1 < j < n$ if T_j is a $\gamma-$interval.

4.3.6 The Two Type Property

Consider the imperfectly matched men with types in a $\phi-$interval T_i but below the watershed x_i^*; i.e. in the open interval (t_{i-1}, x_i^*). They are matched with women all of whom have types less than t_{i-1}, so that the matching of this set of men and women displays NAM. For a man with a type sufficiently close to t_{i-1}, his partner's type is in the adjacent $\gamma-$interval T_{i-1} and close to t_{i-1}. As male type increases from t_{i-1} to x_i^*, the partner's type falls, possibly lying in successively lower $\gamma-$intervals and thus inducing discontinuities in the relationship between male and female types. However, there can only be a finite number of discontinuities (as n is finite); furthermore, this relationship is strictly decreasing – a flat segment would imply a positive mass of men matching with a single type of woman. A similar line of reasoning applies to men with types in the interval (x_i^*, t_i), and to other $\phi-$intervals.

Thus, for almost every t in the support of ϕ^*, there are at most two possible female types that

a man of type t might match with: either he is perfectly matched, or he is imperfectly matched to a woman with a type identified by the relevant NAM submatching as described in the previous paragraph. Similarly, for almost every t in the support of γ^* there are only two possible male types that a woman of type t might match with: type t and one other. We therefore have the *Two Type Property* (TTP):

> **Proposition 5**: If θ is an output maximising matching, then there exists a bijective function ψ with domain $supp(\phi^*)$ and image $supp(\gamma^*)$ such that if $(x, y) \in supp(\theta)$ then either $x = y$ or $y = \psi(x)$.

The function ψ describes that part of the overall equilibrium matching where agents are imperfectly matched and it embodies the principles derived above.[15] It thus consists of N NAM submatchings, each between a $\phi-$ interval and a $\gamma-$ interval, $n-1$ of which are adjacent submatchings where the matches are good but not perfect, and others where the gap between partners' types is large, possibly as great as $\bar{t} - \underline{t}$. Note that there is a finite number of male types (and thus a zero mass of men) for whom in equilibrium, in addition to their own type, there are two further possible female types, corresponding to a discontinuity referred to above, where a change occurs from one NAM submatching to another.

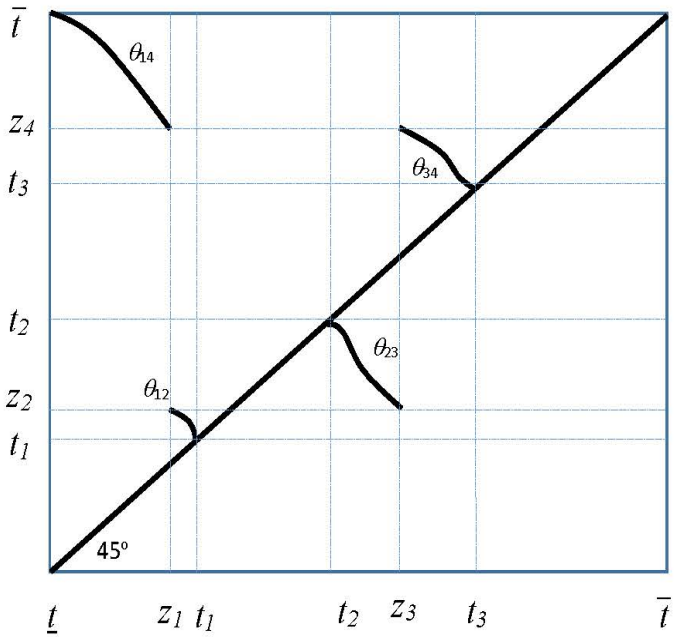


Figure 8: the Two Type Property. The support of the equilibrium matching is shown in bold.

Figure 8 illustrates the Two Type Property when $n = N = 4$ and T is the support of both ϕ and γ. It shows $\phi-$intervals T_1 and T_3 and $\gamma-$intervals T_2 and T_4. Men switch between θ_{14} (the

[15] The function ψ is not quite unique. There may be a finite number of discontinuities where imperfectly matched men in a given $\phi-$ interval, for example, switch from one $\gamma-$interval to another. At the precise point of discontinuity (which is of measure zero) a man may match imperfectly with either of the two female types. We can ensure ψ is single valued by arbitrarily assigning, at each discontinuity, one of the two female types.

only non-adjacent submatching) and θ_{12} at z_1; at z_2 women switch between θ_{12} and θ_{23}; at z_3 men switch between θ_{23} and θ_{34}; and at z_4 women switch between θ_{34} and θ_{14}.[16]

4.4 Payoffs when the type distributions overlap

We now pursue the implications of the principles developed above for the structure of the payoff functions u and v. It is perhaps the Two Type Property that has the most dramatic consequences. Proposition 5 requires that u and v sustain what are in effect two sub-equilibria, which combine to form the overall equilibrium. In the first, we have a subpopulation where both sides have type density κ where $\kappa(t) = \min\{\phi(t), \gamma(t)\}$. Agents choose a partner of their own type, so that

$$\begin{aligned} u(x) &= f(x,x) - v(x) \\ v(y) &= f(y,y) - u(y) \end{aligned}$$

In the second sub-equilibrium, we have a subpopulation with type densities ϕ^* and γ^*. Agents' choices are now given by the function ψ, so that

$$\begin{aligned} u(x) &= f(x,\psi(x)) - v(\psi(x)) \\ v(y) &= f(\psi^{-1}(y),y) - u(\psi^{-1}(y)) \end{aligned}$$

Thus u and v are such that $f(x,y) - v(y)$ is maximised at $y = x$ or $y = \psi(x)$ for any $x \in supp(\phi^*)$ but only at $y = x$ for $x \in supp(\gamma^*)$. Similarly $f(x,y) - u(x)$ is maximised at $x = y$ or $x = \psi^{-1}(y)$ for any $y \in supp(\gamma^*)$ but only at $x = y$ for $y \in supp(\phi^*)$.

To gain more insight into how the payoff functions sustain the equilibrium when g is convex, consider a population initially with non-overlapping type densities $\widetilde{\phi}$ and $\widetilde{\gamma}$ with supports $[\underline{t}, t_1]$ and $[t_1, \overline{t}]$ respectively. As discussed in Section 4.2, it has a NAM equilibrium, and continuous and convex payoff functions, which we label $\widetilde{u}$ and $\widetilde{v}$, with domains $[\underline{t}, t_1]$ and $[t_1, \overline{t}]$; $\widetilde{u}$ is increasing and $\widetilde{v}$ is decreasing. We now expand the population to get overlapping densities ϕ and γ, both with support $[\underline{t}, \overline{t}]$, where $\phi - \widetilde{\phi} = \gamma - \widetilde{\gamma}$, so the additional density is the same for men and women. Thus the equilibrium matching of this larger population consists of a component where agents are perfectly matched (with densities $\phi - \widetilde{\phi}$ and $\gamma - \widetilde{\gamma}$), and the original NAM matching, with densities $\widetilde{\phi}$ and $\widetilde{\gamma}$. Informally, the ψ function of the new equilibrium is the λ function of the original.

We construct the new payoff functions u and v as follows:

$$u(x) = \begin{cases} \widetilde{u}(x) & \text{for } x \in [\underline{t}, t_1] \\ g(0) - \widetilde{v}(x) & \text{for } x \in [t_1, \overline{t}] \end{cases} \tag{16}$$

$$v(y) = \begin{cases} g(0) - \widetilde{u}(y) & \text{for } y \in [\underline{t}, t_1] \\ \widetilde{v}(y) & \text{for } y \in [t_1, \overline{t}] \end{cases} \tag{17}$$

Note (i) u is increasing, and convex for $x \in [\underline{t}, t_1]$ and concave for $x \in [t_1, \overline{t}]$; v is decreasing, and

[16] In this example, $\underline{t} = x_1^* < z_1 < t_1 < z_2 = y_2^* < t_2 < z_3 = x_3^* < t_3 < z_4^* < y_4^* = \overline{t}$.

concave for $y \in [\underline{t}, t_1]$ and convex for $x \in [t_1, \overline{t}]$; (ii) $u(z) + v(z) = g(0)$ for all $z \in [\underline{t}, \overline{t}]$.

We now confirm that this is indeed an equilibrium. Consider first the choice of y by a man of type $x \in [\underline{t}, t_1]$ seeking to maximise $f(x, y) - v(y)$. By construction, the best y in $[t_1, \overline{t}]$ is $\psi(x)$, yielding a payoff of $\widetilde{u}(x)$. This is just the optimal choice of an x type man in the initial equilibrium; in particular it is strictly worse to choose t_1 over $\psi(x)$. For $y \in [\underline{t}, t_1]$ the maximand is $f(x, y) - g(0) + \widetilde{u}(y)$. For $y < x$, both $f(x, y)$ and $\widetilde{u}(y)$ are increasing in y, so we can ignore $y \in [\underline{t}, x)$, and focus on $y \in [x, t_1]$. Over this interval $f(x, y)$ is decreasing and convex in y and $\widetilde{u}(y)$ is increasing and convex, implying that we need only consider $y = x$ and $y = t_1$ as possible choices. The former yields a payoff of $\widetilde{u}(x)$; and the latter yields less than $\widetilde{u}(x)$, since it is strictly worse to choose t_1 over $\psi(x)$. Consider now the choice of y by a man of type $x \in [t_1, \overline{t}]$. For $y < x, f(x, y)$ is increasing and v is decreasing in y, so this leaves $y \in [x, t_1]$. For $x < y \leq t_1$, both $f_y(x, y)$ and $v'(y)$ are negative; however, since $v'(y) = f_y(\psi^{-1}(y), y)$ and $\psi^{-1}(y) < x < y$, then $f_y(x, y) - v'(y) < 0$. Thus the unique optimal choice is $y = x$, giving a payoff of $u(x) = g(0) - v(x)$. A similar argument applies to the choice of x by a woman of type y seeking to maximise $f(x, y) - v(y)$. For $y \in [\underline{t}, t_1]$,the optimal choice is $x = y$, and for $y \in [t_1, \overline{t}]$ a maximum is attained by $x = y$ or $x = \psi^{-1}(x)$.

One way to view the result above is that (16) and (17) exploit to the full the variation in the functions u and v permitted by conditions (11) and (12). Consequently, not only do u and v support same-type partnerships for all agents, but for those on the long or denser side of the market, there is an equally good alternative, as the reasoning above in effect shows that (11) and (12) hold with equality for $t \neq t'$ if and only if either $t = \lambda(t')$ or $t' = \lambda(t)$.

Example (i) extended We take the equilibrium in Section 4.2.1 above with x and y initially uniformly distributed on $[0, 1]$ and $[1, 2]$, and add the same uniform density to both sides. Now

$$\phi(x) = \begin{cases} 2 & \text{for } 0 \leq x < 1 \\ 1 & \text{for } 1 \leq x < x \end{cases} \quad \text{and} \quad \gamma(y) = \begin{cases} 1 & \text{for } 0 \leq y < 1 \\ 2 & \text{for } 1 \leq y < 2 \end{cases}$$

In the new equilibrium matching, a subpopulation with unit densities on $[0, 1]$ for men and $[1, 2]$ for women respectively matches with NAM (i.e. $\psi(x) = 2 - x$); the remainder match like with like, both sides having unit density on $[0, 2]$. This equilibrium is sustained by payoffs

$$u(x) = \begin{cases} u_1 - 2^{\beta-1}(1-x)^\beta & \text{for } 0 \leq x \leq 1 \\ u_1 + 2^{\beta-1}(x-1)^\beta & \text{for } 1 \leq x \leq 2 \end{cases}$$

$$v(y) = \begin{cases} v_1 + 2^{\beta-1}(1-y)^\beta & \text{for } 0 \leq y \leq 1 \\ v_1 - 2^{\beta-1}(y-1)^\beta & \text{for } 1 \leq y \leq 2 \end{cases}$$

where $u_1 + v_1 = \alpha$. Figure 9 shows u and v for $\alpha = 5$ and $\beta = 0.5$, with $u_1 = 2$ and $v_1 = 3$.

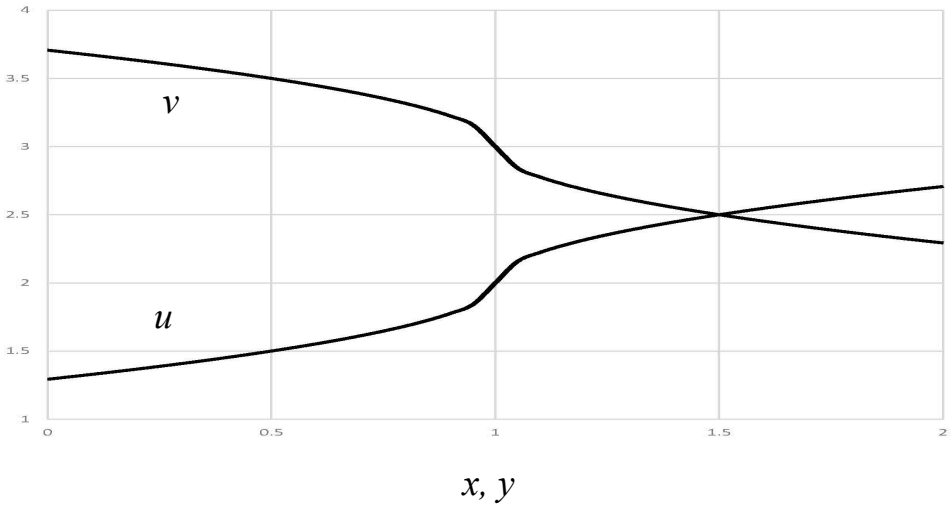
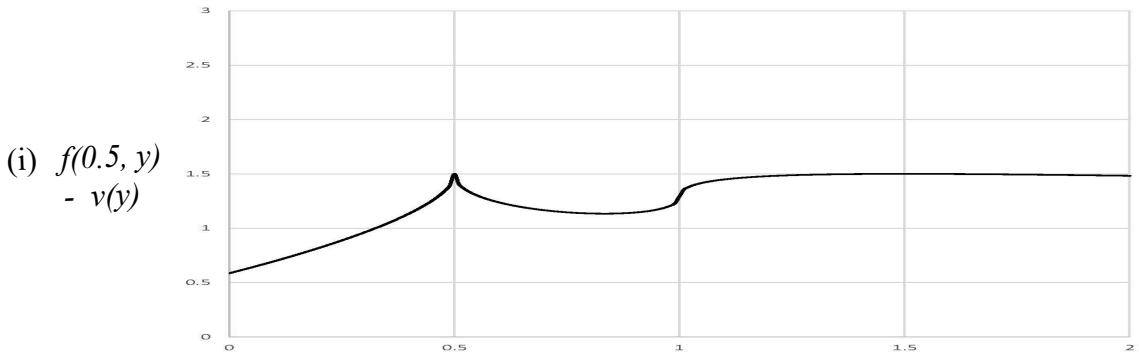


Figure 9: payoffs in example (i) extended



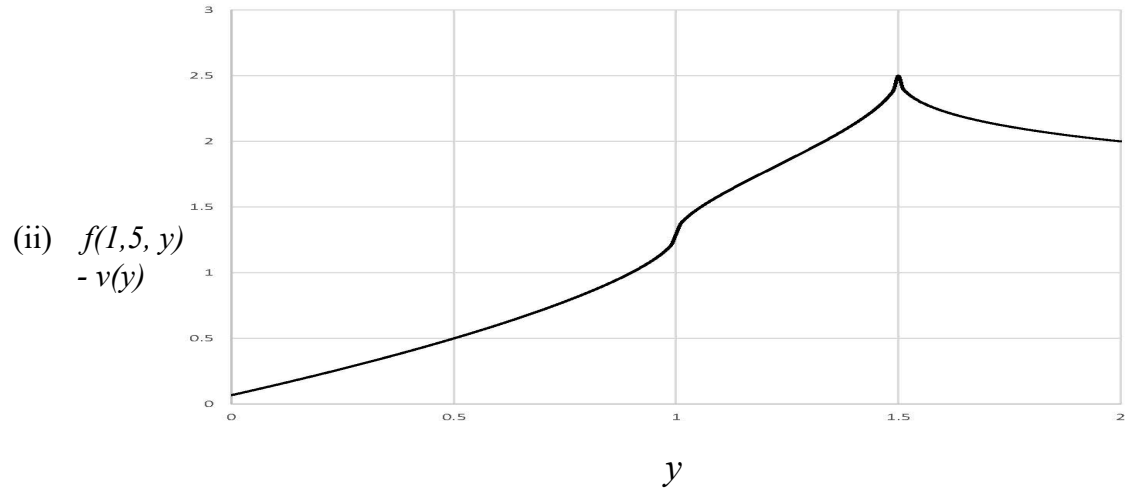


Figure 10: choice of y to maximise $f(x, y) - v(y)$ when (i) $x = 0.5$ (ii) $x = 1.5$

Figure 10 demonstrates the working of the Two Type Property. Panel (i) shows the choice facing a man of type 0.5: $u(0.5)$ can be attained by choosing partners of type 0.5 or 1.5; Panel (ii) shows that for for a man of type 1.5, $u(1.5)$ can be attained only by choosing a partner of type 1.5.

4.4.1 Construction of u and v in the general case

We can apply the reasoning above to each of the N NAM submatchings of positive measure, whether they are adjacent or not.[17] Suppose a NAM submatching θ_{ij} can be sustained as an equilibrium outcome by payoff functions $\widetilde{u}_{ij}$ and $\widetilde{v}_{ij}$, with domains M_{ij} and W_{ij} respectively, bearing in mind that we have a degree of freedom, since $\widetilde{u}_{ij}+c$ and $\widetilde{v}_{ij}-c$ also sustain θ_{ij} as an equilibrium. We then construct u_{ij} and v_{ij} so that

$$u_{ij}(x) = \begin{cases} \widetilde{u}_{ij}(x) & \text{for } x \in M_{ij} \\ g(0) - \widetilde{v}_{ij}(x) & \text{for } x \in W_{ij} \end{cases}$$

$$v_{ij}(y) = \begin{cases} g(0) - \widetilde{u}_{ij}(y) & \text{for } y \in M_{ij} \\ \widetilde{v}_{ij}(y) & \text{for } y \in W_{ij} \end{cases}$$

For each NAM submatching θ_{ij} we thus have a pair u_{ij} and v_{ij}, both with domain $D_{ij} = M_{ij} \cup W_{ij}$.

The functions u_{ij} and v_{ij} are uniquely defined (up to a constant).[18] Hence over the set D_{ij} the payoff functions u and v must equal u_{ij} and v_{ij} respectively; otherwise some agents with types in D_{ij} will not choose partners in accordance with Proposition 5. The union of the domains D_{ij} across all N submatchings is the type space T, so we have all the ingredients to construct u and v. We must do this in a way that (i) ensures the continuity of u and v; and (ii) allows $u+c$ and $v-c$ also to be equilibrium payoff functions.

To do this, we take advantage of the N of the degrees of freedom afforded by the N pairs of functions u_{ij} and v_{ij}. If all N submatchings are adjacent (i.e. $N = n-1$), this is straightforward: each domain D_{ij} is an interval, with $j-i=1$, so denoting by z_i the boundary between $D_{i-1,i}$ and $D_{i,i+1}$ (where agents with types in T_i switch from $\theta_{i-1,i}$ and $\theta_{i,i+1}$) we need to use one degree of freedom to set $u_{i-1,i}(z_i) = u_{i,i+1}(z_i)$, and thus $v_{i-1,i}(z_i) = v_{i,i+1}(z_i)$. Then there are $N-1$ such boundaries, using up $N-1$ degrees of freedom, leaving just one to reflect the indeterminacy in u and v.

If not all NAM submatchings are adjacent (i.e. $N > n-1$) we can still use $N-1$ degrees of freedom to stitch together the N payoff functions u_{ij} and v_{ij}. However, the domain D_{ij} corresponding to a non-adjacent submatching θ_{ij} is not an interval: there is a gap between M_{ij} and W_{ij}. It might thus appear that we have too many points at which to ensure continuity, the lower and upper boundaries of both M_{ij} and W_{ij}, and not enough degrees of freedom. But these boundaries are optimal: the points at which agents switch from one submatching to another, embodied in the graph of the function ψ, gives the output maximising matching. As we now show, this optimality is sufficiently informative to establish continuity.

A Simple Example We return to the case in Figure 8. There are four boundaries or switchpoints z_1, z_2, z_3, and z_4. In constructing u (with $v = g(0) - u$), we can ensure conti-

[17] The argument at the beginning of Section 4.4 assumes type densities $\widetilde{\phi}$ and $\widetilde{\gamma}$ with supports $[\underline{t}, t_1]$ and $[t_1, \overline{t}]$ respectively, in accordance with Assumption 1. But it is straightforward to extend the reasoning to a case of supports $[\underline{t}, t']$ and $[t'', \overline{t}]$ where $t' < t''$, which can then be applied to non-adjacent submatchings.

[18] This is since $d\widetilde{u}_{ij}/dx = f_x(x, \psi(x))$ and $d\widetilde{v}_{ij}/dy = f_y\left(\psi^{-1}(y), y\right)$, so only $\widetilde{u}_{ij}$ and $\widetilde{v}_{ij}$ (plus or minus c) will sustain θ_{ij}, and the two type property requires $u_{ij} + v_{ij} = g(0)$.

nuity at z_1, z_2, and z_3 by using three degrees of freedom of the four available and set $u_{14}(z_1) = u_{12}(z_1), u_{12}(z_2) = u_{23}(z_2)$, and $u_{23}(z_3) = u_{34}(z_3)$. This leaves one degree as required, to reflect the indetermacy in u and v. To ensure continuity at z_4, we use the optimality of the four switchpoints. More precisely,

Proposition 6 For $n = N = 4$, with optimal switchpoints z_1, z_2, z_3 and z_4

$$f(z_1, z_4) + f(z_3, z_2) = f(z_1, z_2) + f(z_3, z_4). \tag{18}$$

Thus NAM and PAM give the same output at the switchpoints, the result of a balance between homogamy pushing the economy towards PAM and Jensen's inequality for convex functions pulling in the opposite direction. [19].

θ_{12} matches shorter men with taller women; in particular it matches men of type z_1 with women of type z_2, and they share the resulting output, so that $u_{12}(z_1) + v_{12}(z_2) = f(z_1, z_2)$. But by construction $u_{12}(z_2) + v_{12}(z_2) = g(0)$; thus

$$u_{12}(z_2) - u_{12}(z_1) = g(0) - f(z_1, z_2) \tag{19}$$

Similarly,

$$u_{23}(z_3) - u_{23}(z_2) = f(z_3, z_2) - g(0); \tag{20}$$

$$u_{34}(z_4) - u_{34}(z_3) = g(0) - f(z_3, z_4); \tag{21}$$

$$u_{14}(z_4) - u_{14}(z_1) = g(0) - f(z_1, z_4). \tag{22}$$

With continuity at z_1, z_2, and z_3,(so that $u_{14}(z_1) = u_{12}(z_1), u_{12}(z_2) = u_{23}(z_2)$, and $u_{23}(z_3) = u_{34}(z_3)$), and using the result of Proposition 6, equations (19) to (22) imply that $u_{34}(z_4) = u_{14}(z_4)$. We therefore need only three of the four available degrees of freedom to stitch together u_{12}, u_{23}, u_{34}, and u_{14} to construct continuous payoff functions u and v, with one degree of freedom left as required.

More complex configurations of NAM submatchings This approach can be extended to accommodate any equilibrium pattern of matching. A full analysis of the general case is given in Appendix B, which shows how, when g is convex, matched agents form into hierarchies of self-contained subpopulations or strata. The boundaries within and between the strata are optimal; together with $N - 1$ degrees of freedom from the N available, this is sufficiently informative to establish the continuity of u and v, with exactly one degree of freedom remaining as required.

[19] Note that as $f(z_1, z_4)$ is less than both $f(z_1, z_2)$ and $f(z_3, z_4)$, (ccc) implies $f(z_3, z_2)$ is greater than both. Thus $z_3 - z_2$ is less than both $z_4 - z_3$ and $z_2 - z_1$, so the middle adjacent submatching, θ_{23}, covers a narrower span of types than θ_{12} and θ_{34}.

4.5 Examples using $f(x,y) = \alpha - |x-y|^\beta$ with $\beta < 1$

4.5.1 Example (ii) revisited

When $\phi(x) = 1$ for $x \in [1,2]$ and $\gamma(y) = 1/3$ for $y \in [0,3]$, the two distributions have an overlap of density $\frac{1}{3}$ on the $\phi-$interval $[1,2]$, giving us the extent of like matching with like. This leaves a subset of men with types distributed on $[1,2]$ with density$\frac{2}{3}$ to be matched with women with types distributed on $[0,1]$ and $[2,3]$ with density $\frac{1}{3}$. To avoid a Category A inefficiency, the women with types in $[0,1]$ must match with the men with types in $[1,1.5]$, and the women with types in $[2,3]$ must match with the men with types in $[1.5,2]$; and to avoid a Category B inefficiency, both of these submatchings must exhibit NAM. This gives

$$\psi(x) = \begin{cases} 3-2x & \text{for } x \in [1,1.5] \\ 6-2x & \text{for } x \in [1.5,2] \end{cases}$$

To sustain this outcome as a competitive equilibrium, the payoff functions u and v must be such that for men maximising $\alpha - |x-y|^\beta - v(y)$ and women maximising $\alpha - |x-y|^\beta - u(x)$

$$\begin{aligned} &\text{a man of type } x \in [1,1.5] \text{ chooses either } y = x \text{ or } y = 3-2x \\ &\text{a man of type } x \in [1,1.5] \text{ chooses either } y = x \text{ or } y = 3-2x \\ &\text{a woman of type } y \in [0,1] \text{ chooses } x = 1.5 - y/2 \\ &\text{a woman of type } y \in [1,2] \text{ chooses } x = y \\ &\text{a woman of type } y \in [2,3] \text{ chooses } x = 3 - y/2 \end{aligned}$$

These conditions are satisfied if:

$$u(x) = \begin{cases} u_1 - 3^{\beta-1}(x-1)^\beta & \text{for } 1 \le x \le 1.5 \\ u_1 - 3^{\beta-1}(2-x)^\beta & \text{for } 1.5 \le x \le 2 \end{cases}$$

$$v(y) = \begin{cases} v_1 - (1.5)^{\beta-1}(1-y)^\beta & \text{for } 1 \le x \le 1.5 \\ v_1 + 3^{\beta-1}(y-1)^\beta & \text{for } 1 \le y \le 1.5 \\ v_1 + 3^{\beta-1}(2-y)^\beta & \text{for } 1.5 \le y \le 2 \\ v_1 - (1.5)^{\beta-1}(y-2)^\beta & \text{for } 2 \le y \le 3 \end{cases}$$

where $u_1 + v_1 = \alpha$. See Figure 11, where $\beta = 0.5$.

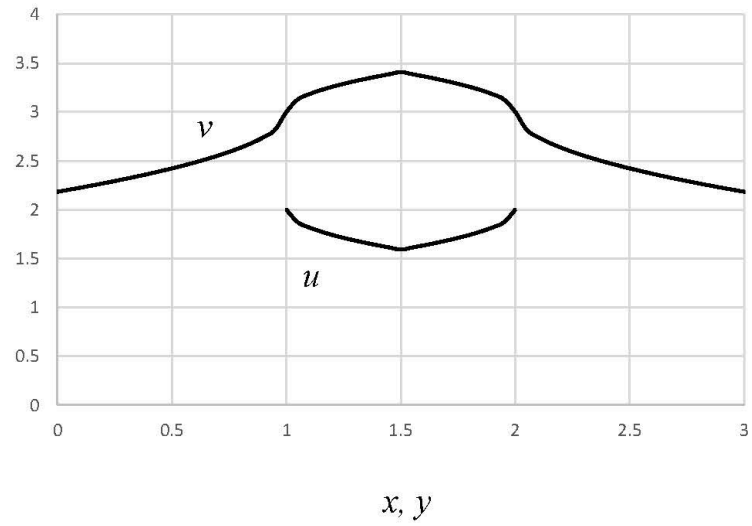


Figure 11: example (ii) with $\beta = 0.5$

4.5.2 Example (iii) revisited and extended

To illustrate how we can determine which NAM submatchings have positive measure, we revisit example (iii) in Section 3.2.3, but with $\beta < 1$. To derive the optimal matching we first apply the principle of maximum matching of like with like. The density of the perfectly matched agents is

$$\kappa(z) = \begin{cases} 2 & 0 \leq z < 1 \\ 1 & 1 \leq z < 3 \\ 2 & 3 \leq z < 5 \\ 1 & 5 \leq z \leq 6 \end{cases}$$

This leaves densities ϕ^* and γ^* of imperfectly matched men and women respectively given by

$$\phi^*(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & 1 \leq x < 3 \\ 0 & 3 \leq x < 5 \\ 1 & 5 \leq x \leq 6 \end{cases} \quad \text{and} \quad \gamma^*(x) = \begin{cases} 1 & 0 \leq y < 1 \\ 0 & 1 \leq y < 3 \\ 1 & 3 \leq y < 5 \\ 0 & 5 \leq y \leq 6 \end{cases}$$

Thus we have γ-intervals, $T_1 = [0, 1]$ and $T_3 = [3, 5]$ and ϕ-intervals $T_2 = [1, 3]$ and $T_4 = [5, 6]$. From Proposition 4, θ_{12}, θ_{23} and θ_{34} have positive measure. To determine whether θ_{14} also has measure measure, we use the analysis of Appendix B. $z_1 \in T_1$ marks the switch between θ_{14} and θ_{12}. The marginal effect of z_1 on agggregate output Q is given by

$$\frac{dQ}{dz_1} = \quad -(6 - 2z_1)^\beta - (2 + 2z_1)^\beta] + (2 - 2z_1)^\beta + (2 - 2z_1)^\beta \tag{23}$$

which is decreasing in z_1 and negative when evaluated at $z_1 = 0$. Thus in equilibrium $z_1 = 0$, and the submatching θ_{14} has zero measure. The resulting matching ψ is given by

$$\psi(x) = \begin{cases} 2 - x & 1 \leq x \leq 2 \\ 6 - x & 2 \leq x \leq 3 \\ 10 - x & 5 \leq x \leq 6 \end{cases}$$

We now axtend this example so that θ_{14} has positive measure. In the interval $[-1, 0]$ we add a density 2 of male types and 3 of female types, and in $[6, 7]$ we add a density 2 of male types and and 1 of female types, so the population remains balanced with $\Omega = 16$. We now have γ-intervals, $T_1 = [-1, 1]$ and $T_3 = [3, 5]$ and ϕ-intervals $T_2 = [1, 3]$ and $T_4 = [5, 7]$. Equation (23) still holds, but with the expanded population z_1 is now bounded below by -1, at which point $\frac{dQ}{dz_1} > 0$.Thus Q is maximised at some z_1^* beween -1 and 0 such that

$$-(6 - 2z_1^*)^\beta - (2 + 2z_1^*)^\beta + (2 - 2z_1^*)^\beta + (2 - 2z_1^*)^\beta = 0. \tag{24}$$

The matching of imperfectly matched agents is now given by

$$\psi(x) = \begin{cases} 2 - x & 1 \leq x \leq 2 - z_1^* \\ 6 - x & 2 - z_1^* \leq x \leq 3 \\ 10 - x & 5 \leq x \leq 6 - z_1^* \\ 6 - x & 6 - z_1^* \leq x \leq 7 \end{cases}$$

and the payoff functions by

$$u(x) = \begin{cases} u_0 + 2^{\beta-1}(3 - x)^\beta & -1 \leq x \leq z_1^* \\ u_1 + 2^{\beta-1}(1 - x)^\beta & z_1^* \leq x \leq 1 \\ u_1 - 2^{\beta-1}(x - 1)^\beta & 1 \leq x \leq 2 - z_1^* \\ u_3 - 2^{\beta-1}(3 - x)^\beta & 2 - z_1^* \leq x \leq 3 \\ u_3 + 2^{\beta-1}(x - 3)^\beta & 3 \leq x \leq 4 + z_1^* \\ u_5 + 2^{\beta-1}(5 - x)^\beta & 4 + z_1^* \leq x \leq 5 \\ u_5 - 2^{\beta-1}(x - 5)^\beta & 5 \leq x \leq 6 - z_1^* \\ u_0 - 2^{\beta-1}(x - 3)^\beta & 6 - z_1^* \leq x \leq 7 \end{cases} \qquad v(y) = \begin{cases} v_0 - 2^{\beta-1}(3 - y)^\beta & -1 \leq y \leq z_1^* \\ v_1 - 2^{\beta-1}(1 - y)^\beta & z_1 \leq y \leq 1 \\ v_1 + 2^{\beta-1}(y - 1)^\beta & 1 \leq y \leq 2 - z_1^* \\ v_3 + 2^{\beta-1}(3 - y)^\beta & 2 - z_1^* \leq y \leq 3 \\ v_3 - 2^{\beta-1}(y - 3)^\beta & 3 \leq y \leq 4 + z_1^* \\ v_5 - 2^{\beta-1}(5 - y)^\beta & 4 + z_1^* \leq y \leq 5 \\ v_5 + 2^{\beta-1}(y - 5)^\beta & 5 \leq y \leq 6 - z_1^* \\ v_0 + 2^{\beta-1}(y - 3)^\beta & 6 - z_1^* \leq y \leq 7 \end{cases} \tag{25}$$

where $u_i + v_i = \alpha$ for $i = 0, 1, 3, 5$, and continuity at $z_1^*, 2 - z_1^*$,and $4 + z_1^*$,is achieved by setting $u_0 = u_1 + 2^{\beta-1}[(1 - z_1^*)^\beta - (3 - z_1^*)^\beta], u_1 = u_3 + 2^{\beta-1}[(1 - z_1^*)^\beta - (1 + z_1^*)^\beta]$,and $u_3 = u_5 + 2^{\beta-1}[(1 - z_1^*)^\beta - (1 + z_1^*)^\beta]$. The first order condition (24) then implies that $u_5 = u_0 + 2^{\beta-1}[(1 - z_1^*)^\beta - (3 - z_1^*)^\beta]$, thus ensuring continuity at $6 - z_1^*$.

Broadly, there are three effects on the matching pattern of this particular expansion in the population. Firstly, there is more PAM due to the additional overlap between φ and γ. Secondly, as θ_{14} now has positive measure, some couples are badly matched, including those with the largest possible type difference of 8. Thirdly, there are changes to the other NAM submatchings: the submatchings θ_{12} and θ_{34} now include more agents, but θ_{23} has fewer. This has important

consequences for the payoff functions.[20] With θ_{12} and θ_{34} the women are shorter than their partners, and hence v is rising over the now larger intervals $[z_1^*, 2 - z_1^*]$ and $[4 + z_1^*, 6 - z_1^*]$; with θ_{23} the women are taller than their partners, so v is falling over the now smaller interval $[2 - z_1, 4 + z_1]$. In addition, with θ_{14} the women are shorter than their partners, so v is rising over $[-1, z_1^*]$ and $[6 - z_1^*, 7]$. Figure 12 shows v before and after the addition of more short women than short men and more tall men than tall women; $\beta = 0.5$ and $v(3)$ is assumed to be unchanged. $u = g(0) - v$ so the effect on u is the opposite.

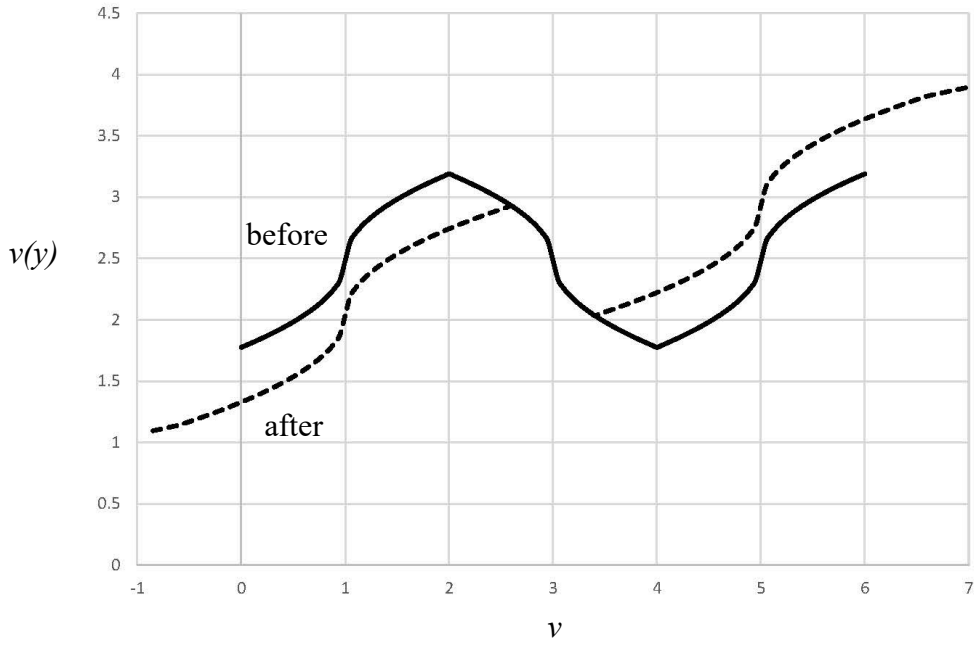


Figure 12: the effect on $v(y)$ in example (iii) when $\beta = 0.5$ of adding more women than men with low types and more men than women with high types. v(3) is assumed to be unchanged.

5 The periodicity of u and v.

Whether g is concave or convex, the payoff functions u and v tend to display a wavelike or periodic structure, despite the very different matching patterns. We now analyse the similarities, and differences, of the two cases in more detail.

We start with g convex, and with $\phi \neq \gamma$. From Section 4.3.5, a $\phi-$interval T_i contains a critical watershed, x_i^* such that men with types in T_i that are less (resp. greater) than x_i^* match with women with types below (resp. above) T_i, implying that u reaches a local minimum, and v a local maximum, at x_i^*. If T_i is a $\gamma-$interval, v reaches a local minimum and u a local maximum at the watershed y_i^*. From Assumption 2, there are n alternating ϕ and $\gamma-$intervals, $T_1, ...T_n$. The watersheds in T_1 and T_n are $\underline{t}$ and $\bar{t}$ respectively, so that u and v both have $n - 2$ turning points (recall that $n \geq 2$ if $\phi \neq \gamma$).

[20] The pre-expansion payoff functions can be read from eq (27), with z_1^* replaced by 0.

In the concave case, the turning points of u and v occur at the $m-1$ points $s_1, ..., s_{m-1}$ where $\Phi(s_i) = \Gamma(s_i)$ and agents are perfectly matched. By Lemma 1, $m < n$, so in comparing the periodicity of the value functions in the concave and convex cases, one critical issue is whether $m = n-1$, a condition that depends only on the two type distributions. If $m < n-1$, then the payoff functions have more turning points in the convex case. But if $m = n-1$, then not only do the two cases have the same number of turning points, but also:

Proposition 7 If $m = n-1$ the turning points of u and v occurs at the same types in the concave case as in the convex case; thus if T_i is a $\phi-$interval then $x_i^* = s_{i-1}$ and if T_i is a $\gamma-$interval $y_i^* = s_{i-1}$.

The following is almost immediate:

Propisition 8 If g is convex, there are no non-adjacent NAM submatchings if and only if $m = n-1$.

5.1 The amplitude of u and v

Some men have a higher payoff than others, and similarly for women. What determines these differences? That is to say, if u and v are wavelike, what determines the amplitude of the waves? To make a comparison of the concave and convex cases that controls for any differences in periodicity, we assume that $m = n-1$. Applying the analysis above, if T_i is a $\phi-$interval with watershed x_i^*, then T_{i+1} is a $\gamma-$interval with watershed y_{i+1}^*; and $[x_i^*, y_{i+1}^*] = [s_{i-1}, s_i]$, a $\Phi-$interval. Assuming the same distributions ϕ and γ, in both cases u has a local minimum, and v a local maximum, at x_i^*, followed by a local maximum of u and a local minimum of v at y_{i+1}^*. This reiterates the importance of the gender difference in the type densities; for a man (resp. woman) it is bad (resp. good) to be in the middle of a $\phi-$interval at some watershed x_i^*, and good (resp. bad) to be in the middle of a $\gamma-$interval at some watershed y_{i+1}^*.

But what determines the difference $\Delta = u(y_{i+1}^*) - u(x_i^*)$, or equivalently $v(x_i^*) - v(y_{i+1}^*)$? In the concave case, we have PAM, and

$$u(y_{i+1}^*) - u(x_i^*) = \int_{x_i^*}^{y_{i+1}^*} f_x(t, \mu(t))dt = \Delta_{conc} \tag{26}$$

Given the surplus function f , Δ_{conc} is therefore determined by the type distributions via the points x_i^* and $y_{i+1}^*$and the matching function $\mu = \Gamma^{-1}(\Phi)$.[21] In the convex case, we have NAM amongst the imperfectly matched agents, pairing men of type x_i^* with women of type y_{i+1}^*. Thus $u(x_i^*) + v(y_{i+1}^*) = f(x_i^*, y_{i+1}^*)$. But $u(y_{i+1}^*) + v(y_{i+1}^*) = g(0)$,(perfectly matched agents share the largest possible output), so

$$u(y_{i+1}^*) - u(x_i^*) = g(0) - f(x_i^*, y_{i+1}^*) = \Delta_{conv} \tag{27}$$

[21] Note that $\mu(x_i^*) = x_i^*$ and $\mu(y_{i+1}^*) = y_{i+1}^*$; for $x_i^* < t < y_{i+1}^*$, $t < \mu(t)$ and $f_x(t, \mu(t)) > 0$.

which, in contrast to Δ_{conc}, is determined by the type distribution only via the points x_i^* and y_{i+1}^*.

To explore this contrast further, we now consider the effect on Δ_{conc} and Δ_{conv} of three possible changes in the type distributions. In each the functions ϕ, γ, Φ, and Γ are replaced by $\widehat{\phi}$,$\widehat{\gamma}$, $\widehat{\Phi}$, and $\widehat{\Gamma}$, and the equilibrium matching θ by $\widehat{\theta}$.

(i) Replication Suppose $\widehat{\phi} = a\phi$ and $\widehat{\gamma} = a\gamma$, where $a > 0$. Thus $\widehat{\Phi} = a\Phi$ and $\widehat{\Gamma} = a\Gamma$. The economy is effectively replicated, and this can be accommodated by an equiproportionate change in the matching, so that $\widehat{\theta} = a\theta$. Any particular type on one side of the market matches with the same type, or set of types, from the other side, so the payoff functions u and v, which both incentivise and give the outcome of choice of partner, are unchanged, as are Δ_{conc} and Δ_{conv}.

(ii) Balanced change We now add the same integrable function b, with support T, to both ϕ and γ, i.e. $\widehat{\phi} = \phi + b$ and $\widehat{\gamma} = \gamma + b$, Thus $\phi - \gamma = \widehat{\phi} - \widehat{\gamma}$ and $\widehat{\Phi} - \widehat{\Gamma} = \Phi - \Gamma$, so there is no change in n, m,the partitions $(\underline{t}, t_1, ..., t_{n-1}, \overline{t})$ and $(\underline{t}, s_1, ..., s_{m-1}, \overline{t})$, or the watersheds ($x_i^*$ or y_i^*) of each interval T_i. However, away from the points $\underline{t}, s_1, ..., s_{m-1}, \overline{t}$, the function that describes positive sorting does change. μ is defined by $\Phi(x) = \Gamma(\mu(x))$. Let $B(t) = \int_{\underline{t}}^{t} b(s)ds$; PAM is now given by $\widehat{\mu}$, where

$$\Phi(x) + B(x) = \Gamma(\widehat{\mu}(x)) + B(\widehat{\mu}(x)). \tag{28}$$

A change to ϕ and γ will therefore change the pattern of matching under concavity and, via equation (26), the rise or fall in payoffs between the (unchanged) points x_i^* and y_{i+1}^*. Equation (28) implies that, for $x_i^* < x < y_{i+1}^*$, $x < \widehat{\mu}(x) < \mu(x)$.[22] Thus $f(x, \widehat{\mu}(x)) > f(x, \mu(x))$ and $0 < f_x(x, \widehat{\mu}(x) < f_x(x, \mu(x))$. Then defining $\widehat{\Delta}_{conc} = \int_{x_i^*}^{y_{i+1}^*} f_x(t, \widehat{\mu}(t))dt$, we have $0 < \widehat{\Delta}_{conc} < \Delta_{conc}$: adding the same density to both sides results in better matches, and reduces the advantage of those of a relatively scarce type (e.g. men of type y_{i+1}^*) compared to those who type is relatively abundant. (e.g. men of type x_i^*). Extending this analysis to all turning points of u and v, we see that insofar as the payoff functions are periodic or wavelike, their amplitude is reduced.

In the convex case, the forces driving the matching pattern and the payoff functions are very different. In particular, u and v can be constructed from the requirement to match the subpopulation with non-overlapping type densities ϕ^* and γ^*, where $\phi^*(t) = \max(\phi(t) - \gamma(t), 0)$ and $\gamma^*(t) = \max(\gamma(t) - \phi(t), 0)$. Since the functions ϕ^* and γ^* are both unchanged by the addition of b to ϕ and γ, the functions u and v are unchanged, as therefore is Δ_{conv}. This surprising result follows from the fact that when g is convex perfect matching can be sustained by any pair of payoff functions from a set of pairs that have limited variation over the domain T. This set does not depend on ϕ and γ, but it includes the pair, unique up to a constant c, that support the equilibrium choices of the imperfectly matched agents with unchanged densities ϕ^* and γ^*.

[22] Applying the transformation Γ^{-1} to both sides of (28), $\mu(x) - \widehat{\mu}(x) = \Gamma^{-1}B(\widehat{\mu}(x)) - \Gamma^{-1}(B(x))$. Γ^{-1} and B are both increasing, so $\mu(x) - \widehat{\mu}(x)$ has the same sign as $\widehat{\mu}(x) - x$. Thus for $x \neq \mu(x)$ (i.e. for $x \notin \{\underline{t}, s_1, ..., s_{m-1}, \overline{t}\}$ either $\mu(x) > \widehat{\mu}(x) > x$ or $x > \widehat{\mu}(x) > \mu(x)$.

(iii) Increase in the dispersion of types To simplify notation in this section, we set $\underline{t} = 0$. We now make all agents are taller by a factor a, where $a > 1$, "stretching out" the type space by replacing $T = [0, \overline{t}]$ with $\widehat{T} = [0, a\overline{t}]$, so that $\widehat{\Phi}(ax) = \Phi(x)$ and $\widehat{\Gamma}(ay) = \Gamma(y)$. PAM is now defined by the function $\widehat{\mu}$, where $\widehat{\Phi}(x) = \widehat{\Gamma}(\widehat{\mu}(x))$, or equivalently $\widehat{\mu}(ax) = a\mu(x)$. Thus $\widehat{\mu}(ax) - ax = a(\mu(x) - x)$.

If t is a fixed point of μ, then at is a fixed point of $\widehat{\mu}$, so in analysing the amplitude of the payoff functions, the turning points of u at x_i^* and y_{i+1}^* are replaced by turning points of $\widehat{u}$ at ax_i^* and ay_{i+1}^*. We now have

$$\begin{aligned}
\widehat{\Delta}_{conc} &= \widehat{u}(ay_{i+1}^*) - \widehat{u}(ax_i^*) \\
&= \textstyle\int_{ax_i^*}^{ay_{i+1}^*} f_x(t, \widehat{\mu}(t))dt \\
&= a\textstyle\int_{x_i^*}^{y_{i+1}^*} f_x(at, \widehat{\mu}(at))dt \qquad \text{(by substitution)} \\
&> \textstyle\int_{x_i^*}^{y_{i+1}^*} f_x(at, \widehat{\mu}(at))dt \qquad \text{(since } a > 1\text{)} \\
&> \textstyle\int_{x_i^*}^{y_{i+1}^*} f_x(t, \mu(t))dt \\
&= \Delta_{conc}
\end{aligned}$$

where the second inequality arises since, for $x_i^* < t < y_{i+1}^*$, $\widehat{\mu}(at) - at > \mu(t) - t > 0$ and $f_x(x, y)$ is positive and increasing in the type difference $y - x$.

Increased dispersion of the type distributions in the concave case therefore increases the amplitude of the waves of the payoff functions and increases the advantages of being a scarce type compared to an abundant type. This arises for two reasons, corresponding to the two inequalities above: (i) $ay_{i+1}^* - ax_i^* > y_{i+1}^* - x_i^*$, so there is a greater range over which payoffs rise and fall between turning points; (ii) the uniform "stretching" of the population means that, unless perfectly matched, all agents are more distant (in type space) from their partners; concavity implies an increasing marginal cost of mismatch, so the rents to be extracted from a good match are greater, increasing the slope of the payoff functions.

In the convex case, we have a similar result, but the reasoning is much simpler:

$$\widehat{\Delta}_{conc} = g(0) - f(ax_i^*, ay_{i+1}^*) > g(0) - f(x_i^*, y_{i+1}^*) = \Delta_{conv}$$

where the inequality arises since f is a decreasing function of the type difference.

6 Payoffs with outside options

In equilibrium, total output is maximised subject to the constraint of the type distributions of men and women. Thus the payoff functions can be interpreted as measures of the effect on maximised output of the marriage market if we relax or tighten the constraint. If we embed our marriage market in a wider economy where single agents have other opportunities, we can analyse which agents end up married by requiring that equilibrium maximises the aggregate output of the

wider economy, including the marriage market. Payoffs that support this equilibrium will then ensure not only that married agents choose the partners that collectively maximise the output of the marriage market, but also that no married agent is better off single and vice versa.

When *more is always better*, this is straightforward because it is the lower types who are single: unambigously they contribute less to total output in the marriage market and this is reflected in payoff functions which, for matched agents, are always increasing in type. But when *like attracts like*, which agents are matched and which are single is not obvious. The payoff functions are not necesssarily monotonic, and whether an agent contributes more or less than others cannot be deduced merely from knowing his or her type, but depends on both type distributions.

6.1 Payoffs as measures of the contribution of types to total output

Consider a balanced population with equal measures of men and women, all matched in the marriage market, and a supermodular output function f. To calculate the marginal effect on total output if we take out a man of some arbitrary type x_1, suppose his partner of type $y_1 = \mu(x_1)$ now becomes the unpartnered woman. This leaves unchanged the partners of the remaining agents, and output falls by $f(x_1, y_1)$. Alternatively, we could leave in the y_1 type woman, and take out some other woman, with type $y_2 = y_1 + \Delta y$, where Δy is small and positive. In this case we have to repartner a man with type $x_2 = \mu^{-1}(y_2)$, so we match him with the y_1 woman and leave all other pairings unchanged. This preserves PAM for all type pairs below (x_1, y_1) and above (x_2, y_2). The (x_2, y_1) pairing introduces a degree of sub-optimality, depending on the size of Δy. The loss of output is now

$$f(x_1, y_1) - f(x_2, y_1) + f(x_2, y_2)$$

which for small enough Δy can be approximated by

$$f(x_1, y_1) + f_y(x_2, y_2)\Delta y$$

If we leave in the type y_2 woman and take out instead a woman with type $y_3 = y_2 + \Delta y$, and pair $x_3 = \mu^{-1}(y_3)$ with y_2 then the output loss is approximately

$$f(x_1, y_1) + f_y(x_2, y_2)\Delta y + f_y(x_3, y_3)\Delta y$$

Continuing in this way, the consequence of rebalancing the population by taking out a woman of type y_p is a loss of output approximated by

$$f(x_1, y_1) + \sum_{1=2}^{p} f_y(x_i, y_i)\Delta y$$

where $\Delta y = (y_p - y_1)/(p-1)$. If we fix the value of y_p, and let $\Delta y \to 0$ and $p \to \infty$, then the approximation to the loss of output if the marriage market loses a man of type x_1 and a woman

of type y_p converges to

$$\Lambda(x_1, y_p) = f(x_1, y_1) + \int_{y_1}^{y_p} f_y(\mu^{-1}(y_i), y_i)dy$$

But the integral is exactly the expression for $v(y_p) - v(y_1)$. Since $f(x_1, y_1) = u(x_1) + v(y_1)$,we have

$$\Lambda(x_1, y_p) = u(x_1) + v(y_p).$$

This general line of analysis does not depend on the particular functional form of f. It is Kantorovich's duality in action, showing how the function $\Lambda(x, y) = u(x) + v(y)$ puts a value on the contribution to the marriage market of a man of type x and a woman of type y, whatever the technology.[23] These two are not necessarily matched in equilibrium: $\Lambda(x, y)$ allows for the rematching of those agents remaining in the market. If x types and y types are not matched in equlibrium then $u(x) + v(y) > f(x, y)$, so $\Lambda(x, y) > f(x, y)$. This may seem surprising, but an alternative interpretation of $\Lambda(x, y)$ is the increase in output if we add an x type man and a y type woman. Then $f(x, y)$ is a lower bound on the additional output, and $u(x) + v(y) - f(x, y)$ is the further gain due to rematching.

6.2 Equilibrium with payoffs to being single

Suppose now that single men and single women get payoffs of $\overline{u}$ and $\overline{v}$ respectively. With these two additional pieces of information we can determine who is matched and who is single, and also resolve the indeterminacy in u and v. Conceptually, we can easily incorporate the possibility that some agents are single into our model. Since $g(x - y)$ decreases with the difference $|x - y|$, there is a unique $\overline{d}$ such that $g(\overline{d}) = \overline{u} + \overline{v}$. An equilibrium with single agents is therefore equivalent to one where all agents are matched, with a surplus function $f(x, y)$ such that $f(x, y) = \overline{u} + \overline{v}$ for $|x - y| > \overline{d}$. Couples for whom $|x - y| < \overline{d}$ are actually married, and produce a joint output more than they could get if single; couples for whom $|x - y| > \overline{d}$ are actually single and produce a joint output $\overline{u} + \overline{v}$; if $|x - y| = \overline{d}$, whether the couple are married or single they produce a joint output $\overline{u} + \overline{v}$.[24] As for the indeterminacy in the payoff functions, given that not all agents are married, single men get $\overline{u}$ and single women get $\overline{v}$; this fixes both u and v.

To see how we can determine who is married and who is single, suppose we have an initial equilibrium where $\overline{u}$ and $\overline{v}$ are low and no agent is single. If we increase $\overline{u}$ or $\overline{v}$, to restore equilibrium requires that we remove from the marriage market any couple, not necesarily matched to each other, such that $u(x) + v(y) < \overline{u} + \overline{v}$; aggregate output then rises by $\overline{u} + \overline{v} - \Lambda(x, y)$. Thus, we remove the agents with low payoffs, i.e. men with types where u is at a minimum and women with types where v is at a minimum. As we have seen in both the concave and convex cases, agents with low payoffs are those with types in relative abundance. But if we remove all the men with types at or around some point t where u is at a minimum and $\gamma(t) > 0$, we will produce

[23] Note that $\Lambda(x, y)$ does not give the contribution of an x type man in isolation. If a man leaves the marriage market, then it becomes unbalanced and we have more women than men. $\Lambda(x, y)$ measures the loss of output if balance is restored by the departure from the market of a y type woman.

[24] Although this interpretation violates Assumption 1 (as $g(|x-y|)$ is not decreasing for $|x-y| > \overline{d}$) it nevertheless satisfies the conditions in Galichon (Propositions 2.2 and 2.3, and Theorem 7.6).

a relative scarcity of men with types at or near t; this would create a local maximum of u at t, implying that total output could be increased by returning to the marriage market some men of type t and making other men single. Thus we should remove relatively abundant male types up to, but not beyond, the point where they are no longer relatively abundant, and similarly for female types.

As agents move from the marriage market to being single, the type distributions of the remaining agents change, which may require the remaining agents to repartner. If an (x, y) pair are initially married and become single, then $\Lambda(x, y) = f(x, y)$; there is thus no rematching of the remaining married agents. But if the (x, y) pair are not initially married, then $\Lambda(x, y) > f(x, y)$ and the remaining married agents must rematch to re-establish optimality.

6.2.1 Which types are married and which remain single? A simple example

To fix ideas, we return to example (ii), where $\phi(x) = 1$ for $x \in [1, 2]$ and $\gamma(y) = 1/3$ for $y \in [0, 3]$, with the joint output of married couples given by $f(x, y) = \alpha - |x - y|^\beta$. If all agents are married, then whether β is greater or less than 1, u has a minimum and v a maximum at 1.5, and v has minima and u maxima at both 0 and 3. If not all agents are married, i.e. $\overline{u}$ and $\overline{v}$ are binding lower constraints on payoffs, then the arguments above suggest that single men will have types around 1.5, the centre of the male distribution, whereas single women will have types at the extremes of the female distribution. Then the symmetry of the distributions means that we can characterise equilibrium by a single variable τ, whose value depends on $\overline{u} + \overline{v}$. Women with types below τ or above $3 - \tau$ are single; they have a total mass of $2\tau/3$. This is balanced by an equal mass of single men, with types with a uniform density of $2/3$ in the interval $[1.5 - \tau/2, 1.5 + \tau/2]$. Married women's types are thus distributed with density $1/3$ on the interval $[\tau, 3 - \tau]$; married men's types have density 1 on $[1, 1.5 - \tau/2]$, density $1/3$ on $[1.5 - \tau/2, 1.5 + \tau/2]$, and density 1 on $[1.5 + \tau/2, 2]$.

g concave for married couples If $\beta > 1$ then we have PAM amongst the married population, with a matching pattern μ given by

$$\mu(x) = \begin{cases} 3(x-1) + \tau & \text{for } 1 \le x \le 1.5 - \tau/2 \\ x & \text{for } 1.5 - \tau/2 \le x \le 1.5 + \tau/2 \\ 3(x-1) - \tau & \text{for } 1.5 + \tau/2 \le x \le 2 \end{cases}$$

Single agents get $\overline{u}$ or $\overline{v}$, so equilibrium payoff functions that support the partner choices of the married population and the marital status of all agents are

$$u(x) = \begin{cases} \overline{u} + (3 - 2x - \tau)^{\beta}/2 & \text{for } 1 \leq x \leq 1.5 - \tau/2 \\ \overline{u} & \text{for } 1.5 - \tau/2 \leq x \leq 1.5 + \tau/2 \\ \overline{u} + (2x - 3 - \tau)^{\beta}/2 & \text{for } 1.5 + \tau/2 \leq x \leq 2 \end{cases}$$

$$v(y) = \begin{cases} \overline{v} & \text{for } y \leq \tau \\ \overline{v} + 1.5(1 - \tau)^{\beta} - 1.5(1 - \frac{2y+\tau}{3})^{\beta} & \text{for } \tau \leq y \leq 1.5 - \tau/2 \\ a - \overline{u} & \text{for } 1.5 - \tau/2 \leq y \leq 1.5 + \tau/2 \\ \overline{v} + 1.5(1 - \tau)^{\beta} - 1.5(\frac{2y-\tau}{3} - 1)^{\beta} & \text{for } 1.5 + \tau/2 \ \leq y \leq 3 - \tau \\ \overline{v} & \text{for } 3 - \tau \leq y \end{cases}$$

where τ is determined by continuity of v at $1.5 - \tau/2$ and $1.5 + \tau/2$; thus $\overline{u} + \overline{v} = a - 1.5(1 - \tau)^{\beta}$. This shows that τ is an increasing function of $\overline{u} + \overline{v}$, (conditional on not all agents being married).

The effects of changes in $\overline{u}$ or $\overline{v}$ are straightforward to analyse. If $\overline{u}$ increases, then so does τ and more men and women become single. Women initially single still get $\overline{v}$. All women initially married are worse off: some become single and get $\overline{v}$; others stay married (but to different partners) but $v(y)$ falls. Initally single men stay single and are better off; some intially married men become single and are better off; men who remain married are better off as $u(x)$ increases. Similar effects follow if $\overline{v}$ increases, *mutatis mutandis*.[25]

g convex for married couples In the case of example (ii), when $\beta < 1$ and all agents are married (i.e. when the constraints $\overline{u}$ and $\overline{v}$ do not bind) we have a mix of PAM and NAM. In contrast to the situation when $\beta > 1$, the NAM component of the population matches low payoff men (in the middle of the male type distribution) with low payoff women (at the extremes of the female distribution). Thus if we start to remove from the marriage market pairs of agents with the lowest payoffs, these will be married couples. Consequently, there is no change in the matching pattern of the remaining married couples (if there were then the initial equlibrium, as it applied to these couples, would have been suboptimal). Formally, for the (x, y) pair with the lowest value of $\Lambda(x, y)$, and thus the first candidates for removal, we have $u(x) + v(y) = f(x, y) = \Lambda(x, y)$; the same equalities hold for subsequent pairs, and we keep moving them from the marriage market into singlehood up to the point where $\Lambda(x, y) = \overline{u} + \overline{v}$.

The matching pattern of the married population thus has perfect matching for types in the interval $[1, 2]$ and a NAM component given by

$$\psi(x) = \begin{cases} 3 - 2x & \text{for } 1 \leq x \leq 1.5 - \tau/2 \\ 6 - 2x & 1.5 + \tau/2 \leq x \leq 2 \end{cases}$$

[25] In particular, some women will be better off single. Thus the model is consistent with the Beckerian account of increasing divorce in the 1950s and 1960s not as a result of change in divorce legislation (an explanation that is, of course, totally at odds with the Coasian flavour of transferable utiity) but due to improved labour market opportunities for women.

The payoff functions that support these partner choices and the marital status of all agents are

$$u(x) = \begin{cases} \overline{u} + 3^{\beta-1}((\frac{1-\tau}{2})^{\beta} - (x-1)^{\beta}) & \text{for } 1 \leq x \leq 1.5 - \tau/2 \\ \overline{u} & \text{for } 1.5 - \tau/2 \leq x \leq 1.5 + \tau/2 \\ \overline{u} + 3^{\beta-1}((\frac{1-\tau}{2})^{\beta} - (2-x)^{\beta}) & \text{for } 1.5 + \tau/2 \leq x \leq 2 \end{cases}$$

$$v(y) = \begin{cases} \overline{v} & \text{for } y \leq \tau \\ \overline{v} + 1.5^{\beta-1}((1-\tau)^{\beta} - (1-y)^{\beta}) & \text{for } \tau \leq y \leq 1 \\ a - \overline{u} - 3^{\beta-1}((\frac{1-\tau}{2})^{\beta} - (y-1)^{\beta}) & \text{for } 1 \leq y \leq 1.5 - \tau/2 \\ a - \overline{u} & \text{for } 1.5 - \tau/2 \leq y \leq 1.5 + \tau/2 \\ a - \overline{u} - 3^{\beta-1}((\frac{1-\tau}{2})^{\beta} - (2-y)^{\beta}) & \text{for } 1.5 + \tau/2 \leq y \leq 2 \\ \overline{v} + 1.5^{\beta-1}((1-\tau)^{\beta} - (y-2)^{\beta}) & \text{for } 2 \leq y \leq 3 - \tau \\ \overline{v} & \text{for } 3 - \tau \leq y \end{cases}$$

Continuity of v at $y = 1$ implies that $\overline{u} + \overline{v} = a - (1.5(1-\tau))^{\beta}$, defining τ as increasing function of $\overline{u} + \overline{v}$ (simular to but not the same function as in the concave case). In the convex case, $1.5(1-\tau)$ is the type difference of the married couples with the greatest mismatch: $y = \tau$ and $x = 1.5 - \tau/2$, or $y = 3 - \tau$ and $x = 1.5 + \tau/2$, for whom $u(x) = \overline{u}$ and $v(y) = \overline{v}$.

7 Conclusion and discussion

There is a wide range of economic and social situations where it is reasonable to assume that agents are drawn to others of a similar type. If we embed this hypothesis of *like attracts like* in a model of two-sided matching with transferable utility, we arrive at conclusions that differ in many respects from the more standard assumption of *more is always better.*

The pattern of matching and sorting, in particular whether we have PAM or NAM, is driven by factors many of which play no role in the standard model. If the two distributions are the same, then whatever the precise technology we have PAM, as all agents can be perfectly matched. If the two distributions are not the same, then the surplus technology is critical. But while concavity of the output function g is equvalent to supermodularity and generates PAM, convexity is not equivalent to submodularity and leads to NAM only in very special circumstances - when there is no overlap in the supports of the two type distributions.

If the two distributions overlap but are not the same, then in the convex case we have maximum matching of like with exactly like and a two-type property: if a man (for example) of type t is on the long side of the market for his type, so $\phi(t) > \gamma(t)$, then he is indifferent between matching with a woman of type t and one of a different type given by the function $\psi(t)$, whereas a woman of type t optimises only by choosing a man of type t. This property of non-unique choices for a positive mass of agents is an essential feature of the equilibrium, and is not found in the standard model.

That *more is not always better* is reflected in the equilibrium payoff functions u and v. Whatever the technology, u and v are roughly mirror images of each other: where one is increasing the other is decreasing. A maximum of one thus occurs at types where the other is at a minimum,

and at these types agents are perfectly matched. So a man, for example, of type t may be matched to a type t woman but surrounded in T by men with higher payoffs.

7.0.2 Implications for our understanding of matching and sorting

Sorting Perhaps most dramatically, the pattern of sorting - PAM, or NAM, or some mix of the two - by itself tells us almost nothing about the underlying technology or preferences that embody homophily. When *more is better*, PAM implies complementarity of inputs and NAM implies substitutability. But with *like attracts like,* a positive correlation between partners' characteristics could be the result of similar type distributions, or of concavity of the output function g. Any positive correlations that we actually observe may thus be the result of *more is better* plus complementarity of inputs, or *like attracts like* plus concavity of g, or *like attracts like* plus similar type distributions.

Interpreting wage and profit data Similarly, in looking at matches of employment, data on wages and profits must be interpreted in a different way. If x and y are measures of a worker's skill type and a firm's technology type respectively, then an observation of workers with higher wage u employed by firms with higher profit v is consistent with *more is better* plus complementarity of skills and technology, but also with *like attracts like* plus convexity of g. Similarly, if high wage workers are employed by low profit firms, this may be evidence of *more is better* plus substitutability, or *like attracts like* plus concavity of g. Of course, one way to distinguish between *like attracts like* and *more is better* is to look directly at u and v as functions of x and y, if the data is available. If *more is better*, they are both increasing; if *like attracts like*, either they both have increasing and decreasing sections or one is increasing and the other decreasing.

Who is unmatched? The analysis of Section 6 embeds the marriage market in a wider economy in which agents have outside options. It is the types where u or v is at a minimum in the absence of a binding outside option that tell us which agents will be single when the option starts to bind. In contrast to the case where *more is better*, having a type close to the mean type on the other side does not always imply a high payoff, and such agents may well remain single. To continue with the example of employment based on skill and technology types, if there is a wide range of technologies (the density γ has a high spread) and a narrower range of skilled workers (ϕ has a lower spread), but ϕ and γ have the same mean (as for example in Fig 5), then u has a U shape and v an inverted U shape. If workers have a good enough outside option (a high $\overline{u}$) then it is those in the middle of the distribution of x who will not be employed (at least, in the market under consideration), and it is the firms at the extremes of the distribution of y who will fall back on their outside option $\overline{v}$. If we apply this line of reasoning to personal relationships, so x and y are male and female types, and we characterise as *popular* those men with types at or near the mean of γ and those women with types at or near the mean of ϕ, then if ϕ has the same mean as γ but a lower spread, it is the unpopular women (at the extremes of γ) who remain single, but the popular men . This may seem paradoxical, but a marriage market with transferable utility is

brutal in attaching a low value to types in high supply, and there is no point in being a good or even perfect match for some women if many other men are too; far better to have a niche type and to be on the short side of the market.

Perfect matches and marital tensions Homophilic preferences naturally draw similar types together, so it is hardly surprising that we are able to provide an explanation of why married couples have similar political or religious views. But this masks a certain tension within relationships. Suppose T is a space of political views and a higher type means less liberal/more conservative, and consider a couple for whom $x < y$; then if it were possible for types to change (and this takes us well beyond the model) the man would want the woman to be more liberal, and she would want him to be more conservative. Unless $\Phi = \Gamma$, almost all couples will experience such disagreements. In the model these tensions are resolved by an equilibrium in which agents accept that everyone has a price and it is not worth paying for a more compatible partner. However, that a couple are perfectly matched ($x = y$) does not necessarily imply complete marital harmony: either u or v is at a local minimum, so one spouse has accepted that it is worth giving up a lot of utility for his or her ideal partner. Just because a couple are well matched does not mean that both are happy.

8 Appendix A: Proofs

Proof of Lemma 1 (i) If $\phi \neq \gamma$, then $\phi(t) = \gamma(t)$ only at a finite number of points $t \in T$ and $\Phi(s) = \int_{\underline{t}}^{s} \phi(t)dt = \int_{\underline{t}}^{s} \gamma(t)dt = \Gamma(s)$ only at a finite number of points $s \in T$, which uniquely define the partition $(\underline{t}, s_1, ..., s_{m-1}, \overline{t})$.

(ii) If $\Phi - \Gamma$ changes sign over an interval $S_i = [s_{i-1}, s_i]$ then $\Phi(s) - \Gamma(s)$ for some $s \in S$, in contradiction to (i).

(iii) Suppose $T_1 = [\underline{t}, t_1]$ is a ϕ-interval; then since $\int_{\underline{t}}^{s_1} [\phi(t) - \gamma(t)]dts = 0$, we must have $t_1 < s_1$; thus $s_1 \in T_i$ for some $i > 1$. If $s_i \in T_j$ and $s_k \in T_l$ where $i < k$. then $j < l$; i.e. two elements of the partition $(\underline{t}, s_1, ..., s_{m-1}, \overline{t})$ cannot be in the same interval of the partition $(\underline{t}, t_1, ..., t_{n-1}, \overline{t})$. Thus $m < n$. A similar argument applies, *mutatis mutandis*, if T_1 is a γ-interval.

Proof of Proposition 1 (i) If not then for some x, $u(x) > Lim_{t \to x^+}\ u(t)$, in which case for $x' - x$ positive but sufficiently small, all women would prefer a man of type x' to a man of type x. In equilibrium, some women must want to match with type x men.

(ii) Suppose that u is increasing and v is non-decreasing at t. No woman of type less than t would choose a man of type t (it is better to choose a male type that is both closer and cheaper), so a type t man's partner must be a type y woman where $y \geq t$. If $y = t$, then $u(t) + v(y) = f(t, y) = g(0)$, as the couple are perfectly matched; and if v is not decreasing at t, then $u(t - \varepsilon) + v(y - \varepsilon) < g(0)$ for ε sufficiently small but finite and positive, in contradiction to equation (7). Thus we take $y > t$. In equlibrium the man of type t chooses to match with a woman of type y out of a choice set that includes women of type t,implying that $v(y) < v(t)$. Consider now

the type of the partner of type t female. She would not be chosen by a man of type less than t (it is better to choose a female type that is closer and no more expensive) so a type t woman's partner must be a type x man, where (by the same argument as above) $x > t$; since she could have chosen a type t partner, $u(x) < u(t)$. If $t < x \leq y$, then $u(x) + v(y) < u(t) + v(y) = f(t, y) < f(x, y)$; if $t < y \leq x$, then $u(x)+v(y) < u(x)+v(t) = f(x, t) < f(x, y)$. In both cases, $u(x)+v(y) < f(x, y)$, which contradicts (7). Thus, if u is increasing at t then v is decreasing at t. A similar argument shows that if u is decreasing at t then v is increasing at t.

(iii) This follows directly from Assumption 1 and the first order conditions associated with (9) and (10).

Proof of Proposition 2 This follows directly from the argument in the main text, particularly Jensen's inequality applied to convex functions.

Proof of Proposition 3 We employ a slight variation on the argument used to establish Proposition 2. Suppose $T_i = [t_{i-1}, t_i]$ and $T_{i+1} = [t_i, t_{i+1}]$ are ϕ and $\gamma-$ intervals respectively. Consider a man of type $t_i - \varepsilon \in T_i$ matched with a woman of type $y \notin T_{i+1}$, and a woman of type $t_i + \varepsilon' \in T_{i+1}$ matched with a man of type $x \notin T_i$.Then for ε and ε' both positive and sufficiently small this foursome could repartner and increase total output. That is to say

$$g(\varepsilon + \varepsilon') + g(|x - y|) > g(|t_i + \varepsilon' - x|) + g(t_i - \varepsilon - y|) \tag{29}$$

which follows from (15) and the continuity of g.

Proof of Proposition 4 Suppose x_1, x_2 and x_3, are all male types in the $\phi-$interval T_i, where $x_1 < x_2 < x_3$, with imperfect matches of y_1, y_2 and y_3 respectively. Then if y_1 and y_3 are in the same $\gamma-$interval T_j, (so $y_1 > y_3$) then y_2 is also in T_j (and hence $y_1 > y_2 > y_3$). Assume intially that $i < j$; if $y_2 < x_1$,then we have a Category B combination; if $y_2 > x_1$, and either $y_2 < y_3$ or $y_2 > y_1$ we have a Category A combination. This means that the subset of female types in T_j matched by the submatching θ_{ij} is convex i.e. an interval; similarly the subset of male types in T_i matched by θ_{ij} is an interval. A virtually identical argument works if $j < i$.

Proof of Proposition 5 This follows directly from the argument in the main text.

Proof of Proposition 6 If we vary one boundary, say z_1, then to preserve the gender balance of the four submatchings, we must vary the other three. More precisely, if z_1 changes by an infinitesimal amount dz_i then

$$\phi^*(z_1)dz_1 = -\gamma^*(z_2)dz_2 = \phi^*(z_3)dz_3 = -\gamma^*(z_4)dz_4$$

To determine the effect of this change on Q, note first that the submatching θ_{14} gains a mass $\phi^*(z_1)dz_1$ of men of type z_1 and the same gain in the mass of women of type z_4, since $\phi^*(z_1)dz_1 = -\gamma^*(z_4)dz_4$ and z_4 has decreased; the result is an increase in output of $f(z_1, z_4)\phi^*(z_1)dz_1$. Similarly

output from θ_{12} decreases by $f(z_1, z_2)\phi^*(z_1)dz_1$, output from θ_{23} increases by $f(z_3, z_2)\phi^*(z_1)dz_1$, and output from θ_{34} decreases by $f(z_3, z_4)\phi^*(z_1)dz_1$. Thus

$$dQ = \{f(z_1, z_4) - f(z_1, z_2) + f(z_3, z_2) - f(z_3, z_4)\}\,\phi^*(z_1)dz_1 \tag{30}$$

In an equilibrium in which all four submatchings are of positive measure, $\phi^*(z_1) > 0$ and $dQ = 0$; hence $f(z_1, z_4) + f(z_3, z_2) = f(z_1, z_2) + f(z_3, z_4)$.

Proof of Proposition 7 Compare the partition $(\underline{t}, t_1, ..., t_{n-1}, \bar{t})-$ the points at which $\phi = \gamma$ and giving the boundaries of the intervals T_i- with the partition $(\underline{t}, s_1, ..., s_{m-1}, \bar{t})$ $-$ the points at which $\Phi = \Gamma$. Clearly, it is not possible for an interval T_i to contain more than one point from $(\underline{t}, s_1, ..., s_{m-1}, \bar{t})$; if $m = n - 1$, we therefore have $\underline{t} \in T_1, s_1 \in T_2, ..., s_{i-1} \in T_i, ..., s_{m-1} \in T_m, \bar{t} \in T_{m+1} = T_n$. Suppose $T_i = [t_{i-1}, t_i]$ is a $\phi-$interval; it contains s_{i-1} and the watershed x_i^*. If $x_i^* < s_{i-1}$, then there is positive mass, $\int_{x_i^*}^{s_{i-1}} \phi^*(t)dt$, of men with types strictly inbetween x_i^* and s_{i-1}, who under convexity must match with women with types greater than t_i. As $\Phi(s_{i-1}) = \Gamma(s_{i-1})$, there is the same mass of men as women with types greater than s_{i-1}, so if $x_i^* < s_{i-1}$, there must be a positive mass of men with types greater than s_{i-1} who match with women with types less than s_{i-1}. This creates a Category A inefficiency, and cannot occur in equlibrium. A similar argument applies if $x_i^* < s_{i-1}$, so $x_i^* = s_{i-1}$; and the same argument applies. *mutatis mutandis,* if T_i is a $\gamma-$interval, in which case $y_i^* = s_{i-1}$.

Proof of Proposition 8 To see that if $m = n - 1$, then in the convex case there are no non-adjacent NAM submatchings, suppose T_1 is a $\phi-$interval. Then all imperfectly matched women with types in the interval $[t_1, y_2^*] \subset T_2$ are matched with men with types in T_1. But as $s_1 = y_2^*$ these men must comprise all the imperfectly matched men with types in T_1;i.e. $\int_{\underline{t}}^{t_1} \phi^*(t)dt = \int_{t_1}^{y_2^*} \gamma^*(t)dt$. This implies that all agents with types no greater than s_1 are either perfectly matched or matched by the adjacent NAM submatching θ_{12}. This argument can be repeated for the remainder of the population; e.g. those agents with types greater than s_1 and no greater than s_2 are either perfectly matched or matched by the adjacent NAM submatching θ_{23}.

To complete the proof, note that if there are no adjacent NAM submatchings then - almost by definition - $m = n - 1$.

9 Appendix B: Horizontal stratification and payoffs when g is convex.

This Appendix shows in the case where g is convex how matched agents form into self-contained subpopulations or strata.

Suppose a NAM sub-matching θ_{ij} has positive measure; we define the interval $R_{ij} = [r_{ij}^-, r_{ij}^+]$, where r_{ij}^- and r_{ij}^+ are the lowest and highest types respectively matched by θ_{ij}.[26] R_{ij} gives the

[26] And because of NAM, ϕ_{ij} matches r_{ij}^- types with r_{ij}^+ types.

span of θ_{ij}. Then all agents with types in R_{ij} match with partners who also have types in R_{ij}. For those agents actually matched by θ_{ij}, this is true by definition; if they are not matched by θ_{ij} (so θ_{ij} is not an adjacent matching) and only one agent in a matched couple has a type in R_{ij} then we have either a Category A or a Category B inefficiency.

Hence R_{ij} defines a self-contained subpopulation: all agents with types in R_{ij} match with other agents with types in R_{ij}. Some will be perfectly matched; others imperfectly, but with a difference in partners' types bounded above by $r_{ij}^{+} - r_{ij}^{-}$. Each NAM submatching of positive measure is associated with a particular self-contained subpopulation, with its own bound on the imperfection of matches. We can thus think of the population as horizontally stratified, each span or *stratum* comprised of agents matching only among themselves.

An agent may belong to more than one stratum. If so, then one stratum is a subset of another; i.e. two strata cannot overlap - otherwise we have a Category B inefficiency.[27] If θ_{ij} and θ_{kl} are both of positive measure and $R_{ij} \subset R_{kl}$ then we say R_{ij} is *contained in* R_{kl} and θ_{ij} is *encompassed by* θ_{kl}. We now show how sets of strata, are connected in a hierarchical or tree-like structure. This structure has important consequences for our understanding of the mix of PAM and NAM, and of the payoff functions u and v.

Firstly, we describe an interative process for generating sets of strata.

1. We begin with the $n-1$ intervals giving the span of adjacent submatchings, forming the sequence $\sigma_1 = (R_{12}, R_{23}, ..., R_{n-1,n})$. We can think of σ_1 as a partial stratification of the population, covering agents with types in $C_1 = \cup_{R_{ij} \in \sigma_1} R_{ij}$. Note that C_1 itself is typically not an interval.

2. We form the sequence σ_2 from σ_1 as follows: for any $R_{ij} \in \sigma_1$, if θ_{ij} is encompassed by θ_{kl}, and θ_{kl} encompasses only submatchings in σ_1, then we replace R_{ij} with R_{kl}, giving an interim sequence σ_2', with as many elements as σ_1; if R_{kl} appears in σ_2' more than once (because θ_{kl} encompasses more than one adjacent submatching), we delete from σ_2' the second and later appearances of R_{kl} to form σ_2. σ_2 covers agents with types in $C_2 = \cup_{R_{ij} \in \sigma_2} R_{ij}$.

3. We continue in this way, forming σ_{h+1} from σ_h. For any $R_{ij} \in \sigma_h$, if θ_{ij} is encompassed by θ_{kl}, and θ_{kl} encompasses only submatchings in $\cup_{h' \leq h} \sigma_{h'}$, then we replace R_{ij} with R_{kl}, giving an interim sequence σ_{h+1}'; if R_{kl} appears in σ_{h+1}' more than once, we delete from σ_{h+1}' the second and later appearances of R_{kl} to form σ_{h+1}. At stage h, σ_h covers types in $C_h = \cup_{R_{ij} \in \sigma_h} R_{ij}$.

4. If after some stage H no further encompassing of any submatchings is possible, the process stops.

In discussing the properties of this process, the following definitions are useful:

[27] If we take two intervals $R_{ij} = [r_i, r_j]$and $R_{kl} = [r_k, r_l]$ then ϕ_{ij} matches r_i types with r_j types, and ϕ_{kl} matches r_k types with r_l types, so if either $r_i < r_k < r_j < r_l$ or $r_k < r_i < r_l < r_j$ we have a category B inefficiency, leaving as possibilities either $r_i < r_k < r_l < r_j$ or $r_k < r_i < r_j < r_l$; i.e. $R_{kl} \subset R_{ij}$ or $R_{ij} \subset R_{kl}$.

Definition 1 *Let h_{ij} be the earliest stage at which R_{ij} appears in a sequence.*
Let A_{ij} be the union of all the elements of $\sigma_{h_{ij}-1}$ that are contained in R_{ij}; we call these the substrata of R_{ij}.
Let M_{ij} and W_{ij} be the set of male and female types respectively matched by θ_{ij}.

For example, let $n = 13$; then we have 12 adjacent submatchings $\theta_{12}, \theta_{23}, ..., \theta_{12,13}$. Suppose that in addition in equilibrium $\theta_{1,12}, \theta_{25}$, $\theta_{5,12}$ and $\theta_{7,10}$ are each of positive measure (it is simple to verify this creates no Category A or B inefficiencies). Then the process described above works as follows:

1. $\sigma_1 = (R_{12}, R_{23}, ..., R_{12,13})$.
2. θ_{25} encompasses θ_{23}, θ_{34} and θ_{45}; and $\theta_{7,10}$ encompasses θ_{78}, θ_{89} and $\theta_{9,10}$; so $\sigma_2 = (R_{12}, R_{25}$, $R_{56}, R_{67}, R_{7,10}, R_{10,11}, R_{11,12}, R_{12,13})$.
3. $\theta_{5,12}$ encompasses $\theta_{56}, \theta_{67}, \theta_{7,10}, \theta_{10,11}$ and $\theta_{11,12}$; so $\sigma_3 = (R_{12}, R_{25}, R_{5,12}, R_{12,13})$. Although $\theta_{1,12}$ encompasses submatchings whose spans are in σ_1 or σ_2,it also encompasses $\theta_{5,12}$, and $R_{5,12}$ is not in σ_1 or σ_2; thus $R_{1,12} \notin \sigma_3$.
4. $\theta_{1,12}$ encompasses θ_{12}, θ_{25} and $\theta_{5,12}$; so $\sigma_4 = (R_{1,12}, R_{12,13})$.
5. No further encompassing of any submatchings is possible; so $H = 4$.

   More generally, the process described in steps 1 to 4 above has the following properties:
   (i) A_{ij} is the union of an odd number of elements of $\sigma_{h_{ij}-1}$ and is an interval.
   (ii) The process stops after a finite number H of stages.
   (iii) Successive sequences have fewer elements: $\#(\sigma_{h+1}) < \#(\sigma_h)$.
   (iv) Successive sequences cover more and eventually all types: $C_h \subset C_{h+1}$, with $C_H = T$.
   (v) For every submatching θ_{ij} of positive weight, $R_{ij} \in \sigma_h$ for some h.
   (vi) If R_{ij} is the first element in σ_h, then $i = 1$; if it is the last element, then $j = n$; if R_{ij} is the immediate predecessor of R_{kl} in the sequence σ_h, then $j = k$.
   (vii) $R_{ij} = M_{ij} \cup A_{ij} \cup W_{ij}$.

It follows that although there is negative sorting within submatchings, there is a nevertheless a broader pattern of positive sorting: a given sequence σ_h is an ordered set of self-contained strata, so there is PAM beween them. As h increases, more types are included, and eventually all types - property (iv) - but this is less informative, as the PAM is occurring at a higher level of aggregation - property (iii).

Additionally, since these NAM submatchings refer to that part of the overall population where like is not matched exactly with like, the perfect matching arising from the overlap of the two type densities is a further source of PAM. Although non-supermodularity of f prevents full PAM of the whole population, there is still a tendency towards matching of like with like: in some cases exactly so, in others with small differences in type, and in yet others where the match may be far from perfect.

Equilibrium and the optimality of the NAM/PAM mix We now use the fact that in equlibrium the overall matching maximises aggregate output. Consider the internal structure of a typical stratum R_{ij} in a sequence σ_h. R_{ij} consists of the interval A_{ij} (the union of substrata of R_{ij}), enclosed by M_{ij} and W_{ij} (the agent types matched by θ_{ij}), men on one side of A_{ij} and women on the other. What characterises the boundaries between M_{ij}, A_{ij} and W_{ij}, and those within A_{ij} between the substrata of R_{ij}?

To answer this, we adopt some simplifying notation. Suppose R_{ij} has p substrata; p is odd. We write $A_{ij} = [z_1, z_{p+1}]$ and the substrata as $[z_1, z_2], [z_2, z_3], ..., [z_p, z_{p+1}]$. We assume, without loss of generality, that T_i is a $\phi-$interval and T_j is a $\gamma-$interval, so that $M_{ij} = [r_{ij}^-, z_1]$ and $W_{ij} = [z_{p+1}, r_{ij}^+]$. Then z_1 is the point at which men switch from θ_{ij} to the submatching with span $[z_1, z_2]$; z_k is the point at which agents switch between the submatching with span $[z_{k-1}, z_k]$ and that with span $[z_k, z_{k+1}]$;and z_{p+1} is the point at which women switch from the submatching with span $[z_p, z_{p+1}]$ to θ_{ij}. The p substrata are all of positive length, so the points $z_1, ..., z_{p+1}$ are distinct.

Let Q_{ij} denote the output produced by the self-contained subpopulation with types in R_{ij}. In equilibrium, Q_{ij} cannot be increased by any reassignment of partners with that subpopulation; i.e. the switchpoints $z_1, ..., z_{p+1}$ maximise Q_{ij}. We now consider the implication of an out-of-equilibrium variation in the switchpoints from their equilibrium values, keeping r_{ij}^- and r_{ij}^+ constant and also the switchpoints (if any) *within* each of the substrata of R_{ij} (i.e. we are analysing a variation that is feasible but suboptimal). Then the values of the changed switchpoints are connected by the need to ensure that for θ_{ij}, and for each of the submatchings whose spans are substrata of R_{ij}, the mass of men matches equals that of women. If z_1 increases, more men are matched by θ_{ij}, implying a decrease in z_{p+1} to enlarge the set of female types $[z_{p+1}, r_{ij}^+]$. But it also implies that fewer men are matched by the submatching with span $[z_1, z_2]$; with no change in the internal boundaries of that substratum, this can only be accommodated by a decrease in z_2. This is turn implies an increase in z_3, a decrease in z_4, and so on; eventually (as p is odd) we arrive at the decrease in z_{p+1}.

More precisely, if z_i changes by an infinitesimal amount dz_i then

$$\phi^*(z_1)dz_1 = -\gamma^*(z_2)dz_2 = \phi^*(z_3)dz_3 = ... = \phi^*(z_p)dz_p = -\gamma^*(z_{p+1})dz_{p+1} \tag{31}$$

To determine the effect on Q_{ij}, note first that the submatching θ_{ij} has gained a mass $\phi^*(z_1)dz_1$ of men of type z_1 and the same gain in the mass of women of type z_{p+1}, since $\phi^*(z_1)dz_1 = -\gamma^*(z_{p+1})dz_{p+1}$ and z_{p+1} has decreased; the result is an increase in output of $f(z_1, z_{p+1})\phi^*(z_1)dz_1$. The submatching with span $[z_1, z_2]$ has lost the same mass, but of men of type z_1 and of women of type z_2, so that output decreases by $f(z_1, z_2)\phi^*(z_1)dz_1$; similarly, the submatching with span $[z_2, z_3]$ has gained men and women of types z_3 and z_2 respectively, so that output increases by $f(z_3, z_2)\phi^*(z_1)dz_1$. Continuing in this way, the change in Q_{ij} is

$$dQ_{ij} = \zeta(z_1)\phi^*(z_1)dz_1 \tag{32}$$

where

$$\zeta(z_1) = f(z_1, z_{p+1}) - f(z_1, z_2) + f(z_3, z_2) - f(z_3, z_4) + f(z_5, z_4)... + f(z_p, z_{p-1}) - f(z_p, z_{p+1}) \quad (33)$$

Thus Q_{ij} is maximised at a value of z_1 such that $\zeta(z_1)\phi^*(z_1) = 0$. Since z_1 is the point at which men switch from θ_{ij} to the submatching with span $[z_1, z_2]$, then in an equilibrium where these both submatchings have positive measure, $z_1 \in int(T_i)$ and so, by Assumption 2, $\phi^*(z_1) > 0$. Thus, in equilibrium, $\zeta(z_1) = 0$;[28] this implies

$$f(z_1, z_{p+1}) + f(z_3, z_2) + f(z_5, z_4)... + f(z_p, z_{p-1}) = f(z_1, z_2) + f(z_3, z_4) + f(z_5, z_6)... + f(z_p, z_{p+1}) \quad (34)$$

(34) has a clear interpretation. Take a finite population with men of types $z_1, z_3, ..., z_p$ and women of types $z_2, z_4, ..., z_{p+1}$,where $z_k < z_{k+1}$. On the left side of (34), we have total output under partial NAM, whereby the shortest man (with type z_1) is matched with the tallest woman (with type z_{p+1}), with PAM amongst the remaining types; the right side gives output under complete PAM. The function $\zeta(z_1)$ gives the excess of partial NAM over PAM, and (34) says that in equilibrium the internal structure of the stratum R_{ij} can be described by switchpoints at which complete PAM is just balanced by an element of NAM. Given the diminishing marginal loss of output as the type difference between matched agents increases, it is the shortest man and the tallest woman who pair up under partial NAM, with PAM preserved among the remaining agents.

Note that the condition $\zeta(z_1) = 0$ does not contain r_{ij}^- or r_{ij}^+, the endpoints of R_{ij}. This implies that, as long as$\int_{r_{ij}^-}^{z_1} \phi^*(t)dt = \int_{z_{p+1}}^{r_{ij}^+} \gamma^*(t)dt > 0$ (so θ_{ij} matches as many men as women), changes in r_{ij}^- and r_{ij}^+ have no effect on the optimal boundaries of the substrata of R_{ij}.

constructing u and v We now use the analysis of stratifiction to construct continuous equilibrium payoff functions u and v. The process follows that used to construct the strata themselves.

1. We start with the $n-1$ adjacent NAM submatchings, and payoff functions as given in vvv. Their spans are the elements of σ_1.

2. Unless $H = 1$, some elements of σ_1 are contained by elements of σ_2. Suppose θ_{ij} encompasses p adjacent submatchings; then R_{ij} is in σ_2 and has p substrata, the union of which forms the interval A_{ij}. For notational simplicity we label these substrata $[z_1, z_2], [z_2, z_3], ..., [z_p, z_{p+1}]$; and we assume without loss of generality that θ_{ij} matches shorter men with taller women, so that $M_{ij} = [r_{ij}^-, z_1]$ and $W_{ij} = [z_{p+1}, r_{ij}^+]$.[29] We join together, or concatenate, the p men's payoff functions of these adjacent submatchings, and the p women's payoff functions, to form $\widehat{u}_{ij}$ and $\widehat{v}_{ij}$, both with domain A_{ij}.We can ensure that $\widehat{u}_{ij}$ is continuous by using up

[28] Although Q_{ij} is not necessarily a concave function of z_1, it is straightforward to show that $\zeta(z_1)$ is decreasing in z_1, so there can be at most one point at which $\zeta(z_1) = 0$.

[29] If θ_{ij} matches shorter women with taller men, in what follows we can swap women and men, and u and v, and arrive at the same conclusion.

$p-1$ of the degrees of freedom referred to above; by construction $\widehat{u}_{ij} + \widehat{v}_{ij} = g(0)$, so $\widehat{v}_{ij}$ is also continuous.

3. We now concatenate $\widehat{u}_{ij}$ and $\widehat{v}_{ij}$ with u_{ij} and v_{ij} respectively to form u^*_{ij} and v^*_{ij} both with domain $R_{ij} = M_{ij} \cup A_{ij} \cup W_{ij}$. Thus

$$u^*_{ij}(x) = \begin{cases} u_{ij}(x) & \text{for } x \in M_{ij} \\ \widehat{u}_{ij}(x) & \text{for } x \in A_{ij} \\ u_{ij}(x) & \text{for } x \in W_{ij} \end{cases}$$

$$v^*_{ij}(y) = \begin{cases} v_{ij}(y) & \text{for } y \in M_{ij} \\ \widehat{v}_{ij}(y) & \text{for } y \in A_{ij} \\ v_{ij}(y) & \text{for } y \in W_{ij} \end{cases}$$

Note that this implies $u^*_{ij}(t) + v^*_{ij}(t) = g(0)$ for all $t \in R_{ij}$.

4. For given functions u_{ij} and v_{ij} we ensure the continuity of u^*_{ij} (and hence of v^*_{ij}) at z_1 and z_{p+1} in two ways. Firstly, we can use the one remaining degree of freedom regarding $\widehat{u}_{ij}$ to set $\widehat{u}_{ij}(z_1) = u_{ij}(z_1)$.

5. The argument for continuity at z_{p+1} is more involved. Over the interval A_{ij}, $\widehat{u}_{ij}$ changes by $\widehat{u}_{ij}(z_{p+1}) - \widehat{u}_{ij}(z_1)$. We now derive an expression for this change. Consider the NAM submatching encompassed by θ_{ij} with span $[z_{k-1}, z_k]$, and suppose that k is even, so shorter men are matched with taller women; then if a man of type z_{k-1} is matched with a woman of type z_k they share the resulting output so that $\widehat{u}_{ij}(z_{k-1}) + \widehat{v}_{ij}(z_k) = f(z_{k-1}, z_k)$. But by construction $\widehat{u}_{ij}(z_k) + \widehat{v}_{ij}(z_k) = g(0)$; thus

$$\widehat{u}_{ij}(z_k) - \widehat{u}_{ij}(z_{k-1}) = g(0) - f(z_{k-1}, z_k) \qquad \text{if } k \text{ is even}$$

Similarly, if k is odd, the submatching with span $[z_{k-1}, z_k]$ matches shorter women with taller men so $z_k, \widehat{v}_{ij}(z_{k-1}) + \widehat{u}_{ij}(z_k) = f(z_{k-1}, z_k)$. As $\widehat{u}_{ij}(z_{k-1}) + \widehat{v}_{ij}(z_{k-1}) = g(0)$ we now have

$$\widehat{u}_{ij}(z_k) - \widehat{u}_{ij}(z_{k-1}) = f(z_{k-1}, z_k) - g(0) \qquad \text{if } k \text{ is odd}$$

Repeating this for all spans $[z_1, z_2], [z_2, z_3], ..., [z_p, z_{p+1}]$, and recalling that p is odd, we have

$$\widehat{u}_{ij}(z_{p+1}) - \widehat{u}_{ij}(z_1) = g(0) - f(z_1, z_2) + f(z_3, z_2) - f(z_3, z_4) + f(z_5, z_4) ..... + f(z_{p-1}, z_p) - f(z_p, z_{p+1})$$

θ_{ij} itself matches men with types in $[r^-_{ij}, z_1]$ with women with types in $[z_{p+1}, r^+_{ij}]$, with payoffs given by u_{ij} and v_{ij}. Thus $u_{ij}(z_1) + v_{ij}(z_{p+1}) = f(z_1, z_{p+1})$. But $u_{ij}(z_{p+1}) + v_{ij}(z_{p+1}) = g(0)$; so

$$u_{ij}(z_{p+1}) - u_{ij}(z_1) = g(0) - f(z_1, z_{p+1})$$

Given $u_{ij}(z_1) = \widehat{u}_{ij}(z_1)$, continuity of u^*_{ij} at z_{p+1} now requires that $\widehat{u}_{ij}(z_{p+1}) - \widehat{u}_{ij}(z_1) = u_{ij}(z_{p+1}) - u_{ij}(z_1)$; i.e. that

$$g(0) - f(z_1, z_2) + f(z_3, z_2) - f(z_3, z_4) + f(z_5, z_4) ... + f(z_{p-1}, z_p) - f(z_p, z_{p+1}) = g(0) - f(z_1, z_{p+1})$$

But this is just the optimality condition $\zeta(z_1) = 0$. Thus $\widehat{u}_{ij}(z_{p+1}) = u_{ij}(z_{p+1})$, so u^*_{ij} is continuous at z_{p+1}.

There is still one degree of freedom arising from the functions u_{ij} and v_{ij},so if u_{ij} were to be adjusted up and v_{ij} down by a constant c, we would also adjust u^*_{ij} up and v^*_{ij} down by c, maintaining continuity of u^*_{ij} and v^*_{ij}. The end result is that for every R_{ij} in σ_2, we have functions u^*_{ij} and v^*_{ij}, each pair having one degree of freedom, which is then carried into the next stage.

6. Essentially the same process can then be repeated for $\sigma_3, \sigma_4, ..., \sigma_H$,giving functions u^*_{ij} and v^*_{ij} for every R_{ij} (if $R_{ij} \in \sigma_1$ then θ_{ij} is an adjacent submatching and u^*_{ij} and v^*_{ij} equal u_{ij} and v_{ij}). Thus at stage h_{ij} (the first stage that R_{ij} appears in a sequence of strata), we form $\widehat{u}_{ij}$ and $\widehat{v}_{ij}$ y concatenating pairs of utility functions whose domains are the substrata of R_{ij} (the elements of $\sigma_{h_{ij}-1}$ that are contained in R_{ij}). Each of these pairs has one degree of freedom. so by using all but one we can ensure continuity of the functions $\widehat{u}_{ij}$ and $\widehat{v}_{ij}$ over the domain A_{ij}. We then join u_{ij} and v_{ij} with $\widehat{u}_{ij}$ and $\widehat{v}_{ij}$ to form u^*_{ij} and v^*_{ij} with domain $M_{ij} \cup A_{ij} \cup W_{ij}$, the remaining degree of freedom being used to ensure continuity at one end of A_{ij}. Continuity at the other end follows from (34), $\zeta(z_1) = 0$, which embodies the condition that the internal boundaries of R_{ij} (i.e. those between M_{ij} and A_{ij}, between A_{ij} and W_{ij}), and between the substrata of R_{ij}, are optimal.

7. The end result is that for each sequence σ_h, whose elements are intervals of T, we have a sequence of pairs of continuous payoff functions, whose domains are the same intervals. As h increases, these intervals cover more and more, and eventually all, of T. At the last stage, H, we can concatenate the elements (if there are more than one) of the final sequence to arrive at the equilibrium payoff functions u and v, with one degree of freedom left.

Note that these steps are not to be interpreted as a constructive proof of the existence of equilibrium payoff functions u and v, but follow from the duality results of Kantorovic and Galichon and the nature of the equilibrium described in Propositions 2 and 5. Having deduced the key properties of the equlibrium matching pattern - maximum possible matching of like with like, with the remainder of the population in a hierarchical structure of NAM submatchings - then if u and v are to sustain that equilibrium, they must satisfy the constructive process described above.

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